STRUCTURALLY STABLE DIFFEOMORPHISMS

OF

NON-SIMPLY CONNECTED MANIFOLDS

by

Michael Maller

A dissertation submitted for the degree of

Doctor of Philosophy

in the University of Warwick

June, 1978

CONTENTS

Acknowledgments

It is a pleasure to acknowledge the assistance I have received in the preparation of this thesis. My supervisor, Dr. Michael Shub, introduced me to the problems which motivated this work. He has truly been a constant source of ideas and encouragement.

I would also like to thank Dennis Sullivan for many helpful and illuminating conversations. Thanks are also extended to my supervisors at the University of Warwick, David Fowler, Brian Sanderson and Antony Manning.

I am grateful to the Ph. D. program in mathematics of the City University of New York for their hospitality, and to the Research Foundation of CUNY for their financial support.

My typists, Mde. Caban, Nalsey Tinberg, and Leon Lemons cheerfully produced an excellent typescript under difficult circumstances.

I would like to thank my mother, Mrs. Ruth Maller, for her help which made it possible for me to return to graduate school.

My wife, Jenny Whitehead, has read and copied and improved this manuscript at every turn. To her, my special thanks.

Declaration

The work in this thesis, apart from the results attributed to others, is to the best of my knowledge, original.

Abstract

Let M be a compact manifold. This thesis concerns the following problem: what is the minimum topological entropy of a structurally stable diffeomorphism in a component of $\mathrm{Diff}^r(M)$? We generalize results of Shub and Sullivan to the non-simply connected case. Let $\mathcal{H}$ be a handle decomposition of M. Shub and Sullivan define a class of stable diffeomorphisms called fitted with respect to $\mathcal{H}$. For these diffeomorphisms the entropy h(f) can be related to the induced chain map f_* on the chain complex $\mathcal{C}(M,\mathcal{H})$ of the handle decomposition. When $\pi_1(M) = 0$ and dim $M \geq 6$ this leads to a characterization in homology theory of the minimum entropy of fitted diffeomorphisms in a component.

When $\pi_1(M) \neq 0$ it is natural to use the chain complex $\tilde{\mathcal{C}}(M,\mathcal{H})$ on the universal cover, as in the proof of the s-cobordism theorem. However, problems arise using the Whitney cancelling lemma in handles of index 2 and (n-2); one can no longer control extraneous intersections by general position. Our main tool is a strong statement of the Whitney lemma in this context, using the relative homotopy group $\pi_2(M_2, M_1, x_0)$. Using results of Whitehead, chain homotopies can be lifted from the chain complex $\tilde{\mathcal{C}}(M,\mathcal{H})$ to the system of relative homotopy groups and realized there. This reduces our problem to algebra, for diffeomorphisms fitted with respect to a fixed handle decomposition.

In order to generalize this result we need to characterize the algebraic chain complexes over $Z[\pi_1]$ which arise from arbitrary handle decompositions of M. When $\pi_1(M) = 0$ and dim $M \geq 6$ Smale proved this reduces essentially to having the correct integral homology. Using our methods in the extreme dimensions, 1, 2 and (n-1), (n-2), and simple homotopy theory, we obtain an s-cobordism type generalization in the non-simply connected case. Let M and $\mathcal{H}$ be as above, but suppose dim $M \geq 9$. If $\mathcal{C}$ is an appropriate chain complex over $Z[\pi_1]$, we show it is realized by a handle decomposition of M if and only if there exists a chain equivalence $\mathcal{C} \longrightarrow \tilde{\mathcal{C}}(M, \mathcal{H})$ with Whitehead torsion zero.

INTRODUCTION

To motivate our discussion, consider the following problem. (Actually, it is too hard and we will solve a much more restricted one).

Problem I: What is the minimum toplogical entropy of a structurally stable diffeomorphism in an isotopy class?

Topological entropy provides a measure of the complexity of a dynamical system (for definitions see (0.6) below). Roughly speaking, it measures how much a continuous map mixes up the open sets; higher entropy indicates a more complicated orbit structure. This make it interesting to determine the minimum entropy of structurally stable diffeomorphisms in a component of $\mathrm{Diff}^r(M)$.

This problem is closely related to Shub's entropy conjecture. Let $f: M \to M$ be a C^1 diffeomorphism of a compact manifold. Since the homology groups $H_k(M; R)$ are vector spaces over R, we can consider the eigenvalues of $f_* : H_*(M; R) \circlearrowleft$. The entropy conjecture states that the logarithm of the largest absolute value of an eigenvalue of f_* is a lower bound for the entropy, or writing $h(f)$ for the entropy and $s(\quad)$ for the spectral radius

$$h(f) \geq \log s(f_* : H_*(M; R) \circlearrowleft).$$

If we think of eigenvalues as a kind of algebraic recurrence, this is the claim that some but not necessarily all of the recurrence behavior of f is detected by the passage to homology theory. The conjecture is unknown in general but partial results exist [28]. Shub and Williams [17] have proved it when f is an Axiom A- no cycle diffeomorphism, and Manning has shown that even for a continuous map $f: M \to M$, $h(f) \geq \log s(f_{*_1} : H_1(M; R) \circlearrowleft)$ [9]. It follows from Manning's theorem that there is a non-trivial lower bound for entropy on each component of $\mathrm{Diff}^r(M)$, and the result of Shub and Williams makes it seem very likely that the conjecture is true for structurally stable diffeomorphisms. Therefore, Problem I can be thought of as asking how sharp the entropy conjecture is for stable diffeomorphisms.

In 1971, Smale showed how to isotope any diffeomorphism of a compact manifold to a stable one by making it preserve a handle decomposition [21]. Shub and Sullivan [16] refined his procedure, defining a class of structurally stable diffeomorphisms which are called fitted with respect to the handle decomposition. Every diffeomorphism is isotopic to a fitted diffeomorphism (in fact, Shub proved that they are C^0 dense in $\mathrm{Diff}^r(M)$ [18]). The entropy of a fitted diffeomorphism can easily be related to the induced map on the chain groups $C_k = H_k(M_k, M_{k-1}; Z)$. In fact, we can use matrices representing f_* on this level (after taking absolute values) as an algebraic model of the minimum fitted recurrence that can be achieved by an isotopy. This leads to the following problem:

Problem II: What is the minimum entropy of a fitted diffeomorphism in a component of $\mathrm{Diff}^r(M)$?

Shub and Sullivan solve this problem for simply connected manifolds of high dimension (≥ 6), reducing it to the algebraic topology of the manifold and component [16]. Their solution is complete but gives a more complicated lower bound than the entropy conjecture. We have to consider the minimal spectral radius over all algebraic chain mappings representing f_* on the integral chain level. This reduces Problem II to a purely algebraic problem.

In this thesis we will discuss Problem II when $\pi_1(M) \neq 0$. We obtain an analogous reduction to algebra, provided dim $M \geq 9$ and we restrict attention to fitted diffeomorphisms with one source and one sink. In order to state these results we need to review the definition of fitted diffeomorphism.

Let M^n be a compact manifold with $\partial M = \phi$. Recall that a handle decomposition $\mathcal{H}$ of M is a sequence of submanifolds with boundary, $M_0 \subset M_1 \subset . . . . \subset M_n = M$, such that

$$\overline{M_k - M_{k-1}} = \bigcup_{i=1}^{r_k} (D_i^k \times D_i^{n-k}) \text{ and the k-handles } \phi_i^k = (D_i^k \times D_i^{n-k})$$

are attached to ∂M_{k-1} by disjoint embeddings of the $(S_i^{k-1} \times D_i^{n-k})$.

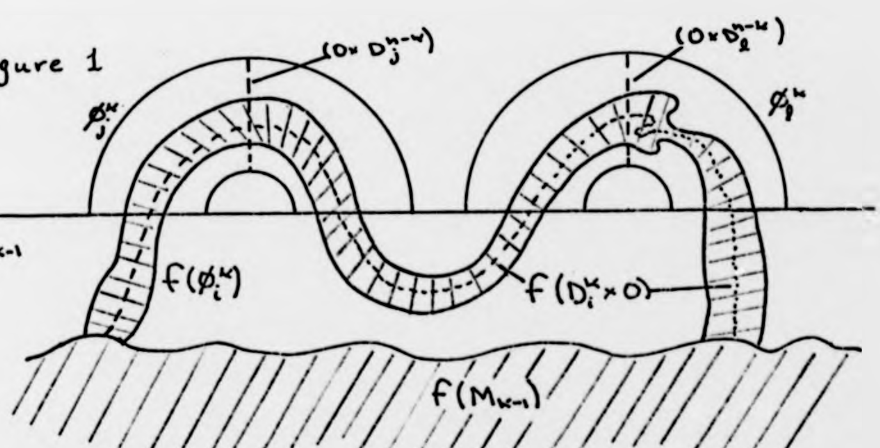
The discs $(D_i^k \times p)$ are called <u>core</u> discs and the $(q \times D_i^{n-k})$ <u>transverse</u> discs of ϕ_i^k.

0.1) <u>Definition</u> Let $f: M \to M$ be a diffeomorphism. We will say $f \in T_{\mathcal{H}}$ if

(i) $f(M_k) \subset \operatorname{int} M_k$

(ii) $f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ $\forall i, j$
$k = 0, \dots n$.

figure 1



Since the dimensions are complementary, $f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ will consist of a finite number of isolated points. We define a <u>geometric</u> <u>intersection</u> <u>number</u> by

$$g_{ij}^k = \text{card.}\ (f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k}))$$

and record these in a geometric intersection matrix $G_k = (g_{ij}^k)$.

If we assign orientations to all the core discs then each point of intersection $p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ receives an orientation number ORE (p) = $\pm$ 1. We can then define an <u>algebraic intersection number</u>

$$a_{ij}^k = \sum_{p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})} \mathrm{ORE}\,(p)$$

Clearly $g_{ij}^k \geq |a_{ij}^k|$. Notice that choosing orientations for the core discs $(D_i^k \times 0)$ yields a geometric basis for the chain groups $C_k = H_k(M_k, M_{k-1}; Z)$. We will write $\mathcal{C}(M, \mathcal{H})$ for this chain complex induced by the handle decomposition. Since $f(M_k) \subset \mathrm{int}\, M_k$, $f_* : \mathcal{C}(M, \mathcal{H}) \circlearrowleft$ is defined, and the matrix of f_{*_k} with respect to the basis of oriented core discs is just $A_k = (a_{ij}^k)$.

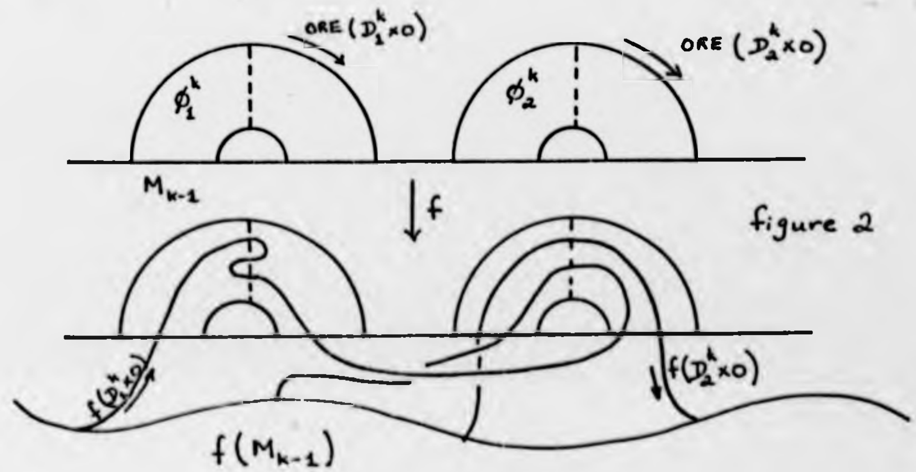


figure 2

In this example G_k is $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and A_k is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

0.2. <u>Definition</u> We will say f is <u>fitted</u> with respect to $\mathcal{H}$ and write $f \in F_{\mathcal{H}}$, if $f \in T_{\mathcal{H}}$ and in addition:

(i) the image of each core disc $f(D_i^k \times p)$ completely contains any core disc it meets

(ii) the image of each transverse disc $f(q \times D_i^{n-k})$ is completely contained in any transverse disc it meets

(iii) f is uniformly expanding in the core directions and uniformly contracting in the transverse directions.

<u>Note</u>. Actually, we need to require that the attaching maps of $\mathcal{H}$ are themselves fitted.

0.3 <u>Theorem</u> (Shub-Sullivan). Let $\mathcal{H}$ be a fitted handle decomposition

1) Every diffeomorphism $f: M \to M$ is isotopic to one in $F_{\mathcal{H}}$. If $f \in T_{\mathcal{H}}$ to begin with then the fitting isotopy does not change the geometric or algebraic intersection numbers.

2) Fitted diffeomorphisms satisfy the Axiom A and strong transversality condition and therefore are structurally stable.

3) If $f \in F_{\mathcal{H}}$ then $h(f) = \max_i \log s(G_i)$.

4) $F_{\mathcal{H}}$ is C^0 dense in $\mathrm{Diff}^r(M)$ $\square$

Suppose $f \in F_{\mathcal{U}}$. Since $g_{ij}^{k}(f) \geq |a_{ij}^{k}(f)|$ it follows from theorems about non-negative matrices [29] that $s(G_i) \geq s(|A_i|) \geq s(A_i)$. On the other hand, the chain map (A_i) induces f_* on homology so $s(A_i) \geq s(f_{*i})$. Then we obtain from part (3) of the theorem:

$$h(f) = \max_i \log s(G_i) \geq \max_i \log s(|A_i|)$$

$$\geq \max_i \log s(A_i)$$

$$\geq \log s(f_* : H_*(M; R) \circlearrowleft)$$

Therefore the fitted diffeomorphisms satisfy the entropy conjecture. In fact, we obtain a finer lower bound. Let $f \in T_{\mathcal{U}}$ so that $f_* : \mathcal{C}(M, \mathcal{U}) \circlearrowleft$ is defined. If f is isotopic to $g \in F_{\mathcal{U}}$ then $g_* : \mathcal{C}(M, \mathcal{U}) \circlearrowleft$ is also defined and the two chain maps are chain homotopic, $f_* \simeq g_*$, while by the inequality above we have $h(g) \geq \log s(|g_*|)$. This gives a lower bound for the entropy of any diffeomorphism in $F_{\mathcal{U}}$ isotopic to f.

0.4 <u>Proposition</u> Let $f \in T_{\mathcal{U}}$, g isotopic to f and fitted with respect to $\mathcal{U}$. Then

$$h(g) \geq \underset{E \simeq f_*}{\text{minimum}} \log s(|E|)$$

where E ranges over all chain maps $E: \mathcal{C}(M, \mathcal{U})\circlearrowleft$ chain homotopic to f_*. □

The same argument applies as well to finer algebraic structures associated to the filtration (M_i) of M. The problem is which algebra records the geometric intersection numbers that can be realized by isotopy.

Shub and Sullivan [16] show that in the simply connected case the appropriate models to use are the chain groups $C_k = H_k(M_k, M_{k-1}; Z)$. This means:

(i) for $f \in T_{\mathcal{U}}$, provided dim $M \geq 5$ and $\mathcal{U}$ has no (1) or (n-1) handles, we can realize $g^k_{ij} = |a^k_{ij}|$ by a filtration-preserving isotopy (i.e. an isotopy f_t $0 \leq t \leq 1$ such that $f_t(M_i) \subset \text{int } M_i$ $\forall t$ $0 \leq t \leq 1$, $\forall_i$ $0 \leq i \leq n$)

(ii) if we relax the requirement that the isotopy preserve the filtration, then algebraic operations on the chain level (i.e. chain homotopies) can be realized by isotopy. This reduces Problem II for a fixed $\mathcal{U}$ to a purely algebraic problem.

Our problem is to find an analogous reduction to algebra when $\pi_1(M) \neq 0$. As in the proof of the s-cobordism theorem, we replace the chain groups C_k with the chain groups on the universal cover $\tilde{C}_k = H_k(\tilde{M}_k, \tilde{M}_{k-1}; Z)$. This works well in the middle dimensions $3 \leq k \leq n-3$. However, problems arise using the Whitney Lemma in dimensions 1, 2 and n-1, n-2. Shub and Sullivan point out we can use the free group $\pi_1(M_1, x_0)$ in dimension 1. In Chapter One we show that the appropriate model in dimension 2 is $\pi_2(M_2, M_1, x_o)$. Then the top dimensions (n-1), (n-2) can be dealt with using the induced maps of (f^{-1}) on the homotopy groups of the dual handle decomposition $\mathcal{H}'$. (We need to assume dim $M \geq 5$ and $\mathcal{H}$ has a single 0-handle and a single n-handle).

There is a complex of relative homotopy groups, related to the chain complex $\tilde{\mathcal{C}}(M, \mathcal{H})$ by the following diagram:

$$\begin{tikzcd}
\to \pi_k(M_k, M_{k-1}, x_o) \arrow[r, "d_k"] \arrow[d, "h_k"] & \cdots \arrow[r, "d_3"] & \pi_2(M_2, M_1, x_o) \arrow[r, "d_2"] \arrow[d, "h_2"] & \pi_1(M_1, x_o) \to 1 \arrow[d, "h_1"] \\
\to H_k(\tilde{M}_k, \tilde{M}_{k-1}) \arrow[r, "\partial_k"] & \cdots \arrow[r, "\partial_3"] & H_2(\tilde{M}_2, \tilde{M}_1) \arrow[r, "\partial_2"] & H_1(\tilde{M}_1, \tilde{M}_o) \arrow[r, "\partial_1"] & H_o(\tilde{M}_o)
\end{tikzcd}$$

The boundary maps (d_i) come from the homotopy exact sequences of the adjacent pairs so $d_{i-1} \circ d_i = 0$. The maps h_k are essentially the Hurewicz homomorphisms for $k \geq 2$ and are isomorphisms for $k \geq 3$. These groups were studied by Whitehead in the case of a CW complex [25]. We will write $\rho(M, \mathcal{H})$ for the sequence of relative homotopy groups following Whitehead, and write $f_\#: \rho(M, \mathcal{H}) \circlearrowleft$ for the induced maps. Since $h_k: \rho_k \to \bar{C}_k$ is an isomorphism for

$k \geq 3$, the essential information we are adding to $\widetilde{\mathcal{C}}(M, \mathcal{H})$ is the presentation of $\pi_1(M)$ given by $\rho_2 \xrightarrow{d_2} \rho_1$. The advantage of using the sequence $\rho\ (M, \mathcal{H})$ is that the appropriate algebraic operations can be defined as chain homotopies on $\rho(M, \mathcal{H})$.

In Chapter Two we prove that a chain homotopy $f_\# \underset{\xi}{=} E$ on $\rho(M, \mathcal{H})$ can be realized by an isotopy of f. The proof is similar to the simply connected case but requires more work in the top dimension. This reduces Problem II for a fixed handle decomposition to the non-trivial but purely algebraic problem of determining the minimum spectral radius in a chain homotopy class.

Thus far, we have been considering diffeomorphisms fitted with respect to a fixed handle decomposition. In order to generalize the reduction to algebra of Problem II, we need to characterize the chain complexes which arise from handle decompositions of the manifold. For simply connected manifolds of high dimension, this reduces to having the correct homology. The following theorem is essentially proved by Smale in [22], but was apparently first stated in this form by Shub [19].

<u>Theorem</u> (Smale). Suppose M is a compact connected manifold, $\partial M = \phi$, $\pi_1(M) = 0$, dim $M \geq 6$. If $(\mathcal{C}, \partial)$ is a chain complex of finitely generated free abelian groups $0 \to C_n \to \cdots \to C_o \to 0$ such that $C_{n-1} = C_1 = 0$ and $H_*(\mathcal{C}; Z) \cong H_*(M, Z)$ then $\mathcal{C}$ is the chain complex of some handle decomposition of M. $\square$

In the non-simply connected case we consider chain complexes over $Z[\pi_1]$. It is too much to expect that a condition on homology should suffice. However, if we start with a given handle decomposition $\mathcal{H}$ we can consider all the ways $\tilde{\mathcal{C}}(M, \mathcal{H})$ can be modified, by isotopies of the attaching maps, trading handles etc., as in the proof of the s-cobordism theorem.

This suggests the condition should be that there exist a chain homotopy equivalence $\mathcal{C} \to \check{\mathcal{C}}(M, \mathcal{H})$ with Whitehead torsion zero. A familiar problem arises using the Whitney lemma in dimensions 1, 2 and n-1, n-2. Using the results of Chapters One and Two this problem can be avoided if the simple chain equivalence is defined on the level of the homotopy groups $\rho(M, \mathcal{H})$.

We obtain the following characterization of the algebraic chain complexes realized by handle decompositions of M. (Technical difficulties in the proof force us to assume dimension M $\geq$ 9 but I believe $\geq$ 6 is actually sufficient).

0.5) Theorem. Let M^n be a connected closed manifold, dimension M $\geq$ 9, and $\mathcal{H}$ a handle decomposition of M. Let $\mathcal{C}$ be a chain complex over $Z[\pi_1]$ such that both $\mathcal{C}$ and its dual complex admit homotopy groups. Then $\mathcal{C}$ is realized by a handle decomposition of M if and only if there exists a chain homotopy equivalence $G : \check{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$ with Whitehead torsion zero. □

Using (0.5) we reduce Problem II to a purely algebraic problem, provided dimension M $\geq$ 9 and we consider fitted diffeomorphisms with a single source and a single sink.

0.6 Background facts and Definitions.

I owe the reader a definition of topological entropy. This concept was introduced by Adler, Konheim and McAndrew for continuous maps of a topological space [1]. The following definition due to Bowen [2] is more convenient for us.

Definition Let X be a compact metric space and $f: X \to X$ a continuous map. A set $A \subset X$ is said to be (n, ε) separated for f if $\forall\, x, y \in A$, $x \neq y$, $\exists\, i$, $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \varepsilon$. That is, each point in A represents, up to tolerance ε, a distinct orbit type of length n. Let $r(f, n, \varepsilon)$ = maximal cardinality of an (n, ε) separated set for f, and $h_\varepsilon(f) = \lim\sup \frac{1}{n} \log r(f, n, \varepsilon)$. We define the topological entropy by

$$h(f) = \lim_{\varepsilon \to 0} h_\varepsilon(f).$$

Thus the entropy measures the asymptotic exponential growth rate of the number of orbit types. □

Let M be a compact manifold, $\partial M = \emptyset$. Two diffeomorphisms $f, g: M \to M$ are said to be topologically conjugate (written $f \sim g$) if there exists a homeomorphism $h: M \to M$ such that $hf = gh$. A diffeomorphism $f: M \to M$ is C^r structurally stable if there is a neighbourhood $N(f)$ in the C^r topology on $\mathrm{Diff}^r(M)$ such that $g \in N(f)$ $\Rightarrow g \sim f$. That is, up to continuous change of co-ordinates the orbit structure is constant in a C^r neighbourhood of f.

A set $\Lambda \subset M$ is invariant for f if $f(\Lambda) = \Lambda$. An invariant set Λ for f has a hyperbolic structure if there is a continuous (Tf) invariant splitting $T_\Lambda M = E^s \oplus E^u$ such that $Tf|_{E^s}$ is uniformly contracting and $Tf|_{E^u}$ is uniformly expanding. More precisely, there exist constants $C > 0$ and $\lambda < 1$ such that

$$\| Tf^n|_{E^s} \| \leq C\lambda^n \text{ for } n > 0$$

$$\| Tf^{-n}|_{E^u} \| \leq C\lambda^n \text{ for } n > 0.$$

The non-wandering set $\Omega(f)$ is the set of all points $x \in M$ such that for all neighbourhoods $N(x)$ $\exists$ $k > 0$ such that $f^k(N) \cap N \neq \phi$. $\Omega(f)$ is a closed invariant set for f and contains all the periodic points of f and all the accumulation points of orbits. In some sense $\Omega(f)$ is the heart of the dynamical system. Bowen proved that $h(f) = h(f|_{\Omega(f)})$ [2].

A diffeomorphism $f: M \to M$ is said to satisfy Smale's Axiom A provided

(a) $\Omega(f)$ has a hyperbolic structure

(b) $\Omega(f)$ is the closure of the periodic points of f.

The stable and unstable manifolds of f are defined by:

$$W^s(x) = \{ y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty \}$$

$$W^u(x) = \{ y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty \}$$

If f satisfies Axiom A then the stable and unstable manifolds are 1-1 immersed copies of Euclidean spaces tangent to E^s and E^u. f is said to satisfy the strong transversality condition if $W^s(x) \pitchfork W^u(y)$ $\forall x, y \in M$.

Theorem (Robbin, Robinson). Let $f: M \to M$ be a C^1 diffeomorphism. If f satisfies Axiom A and strong transversality (A.S.) then f is structurally stable. [14]

A diffeomorphism is said to be Morse Smale (M.S.) if and only if

(1) $\Omega(f)$ is finite

(2) the periodic points are hyperbolic

(3) $W^s(p) \pitchfork W^u(q)$ for $p, q \in \Omega$

Thus the M.S. diffeomorphisms are the A.S. diffeomorphisms with finite Ω. It follows from a theorem of Bowen [2] that if f is A.S. and $h(f) = 0$ then f is M.S.

Let $\mathcal{U}$ be a handle decomposition, $M_o \subset M_1 \subset \ldots \subset M_n = M$. The dual handle decomposition $\mathcal{U}'$ is obtained by turning the manifold upside down. Let $M'_{n-i} = \overline{M - M_{i-1}}$. Then

$$\begin{aligned} \overline{M'_{n-i} - M'_{n-i-1}} &= \overline{M_i - M_{i-1}} \\ &= \cup \phi^i_j \\ &= \cup (D^i_j \times D^{n-1}_j). \end{aligned}$$

We regard $(D_j^i \times D_j^{n-i})$ as an (n-i) handle $\phi_j^{n-i\,\prime}$ with core disc $(O \times D_j^{n-i})$ and transverse disc $(D_j^i \times 0)$.

If $f \in T_{\mathcal{U}}$ then $(f^{-1}) \in T_{\mathcal{U}}{}'$, and if $f \in F_{\mathcal{U}}$ then $(f^{-1}) \in F_{\mathcal{U}}{}'$. $\square$

Finally, throughout this thesis, all manifolds and diffeomorphisms will be C^r, $r \geq 1$.

Notation

M^n	a closed connected C^r $(r \geq 1)$ manifold, of dimension n
$\mathcal{H}$	a handle decomposition
$\mathcal{C}(M, \mathcal{H})$	the cellular chain complex $H_k(M_k, M_{k-1}; Z) \longrightarrow H_{k-1}(M_{k-1}, M_{k-2}; Z)$
$\tilde{\mathcal{C}}(M, \mathcal{H})$	the chain complex on the universal cover $\tilde{M}$ $H_k(\tilde{M}_k, \tilde{M}_{k-1}; Z) \longrightarrow H_{k-1}(\tilde{M}_{k-1}, \tilde{M}_{k-2}; Z)$
$\rho(M, \mathcal{H})$	the homotopy system $\rho_i = \pi_i(M_i, M_{i-1}, x_o)$
$T_{\mathcal{H}}$	a class of diffeomorphisms of M which preserve $\mathcal{H}$
$F_{\mathcal{H}}$	the class of diffeomorphisms of M fitted with respect to $\mathcal{H}$
ϕ_i^k	a k-handle
$(D_i^k \times O)$	the central core disc of ϕ_i^k

$(O \times D_j^{n-k})$	the central transverse disc of ϕ_i^k
Γ_i^k	the base path of ϕ_i^k
$(\Gamma_i^k\|_p)$	Γ_i^k extended by an arc in ϕ_i^k to some point $p \in \phi_i^k$
$[\phi_i^k]$	the corresponding element of $\pi_k(M_k, M_{k-1}, x_o)$
$ORE(\phi_i^k)$	orientation induced by Γ_i^k and the chosen orientation of ϕ^o
$\sigma_{i,c}^k$	orientation of $(D_i^k \times O)$
$\dot{\sigma}_{i,c}^{k-1}$	induced orientation of $\partial(D_i^k \times O) = (S_i^{k-1} \times O)$
$\sigma_{i,t}^{n-k}$	orientation of $(O \times D_i^{n-k})$
$[\sigma, \tau]$	concatenation of orientations
$\langle\ ,\ \rangle$	comparison of orientations
$\langle\ ,\ \rangle_p$	comparison at a point p

f/A	restriction
$A \setminus p, q$	complement
$s(\ )$	spectral radius

CHAPTER ONE

Geometric and Algebraic Intersection Numbers.

The object of this chapter is to prove the following lemma.

1.1 Non-simply connected cancelling lemma.

Let M be a compact connected manifold, $\partial M = \emptyset$, $\dim M \geq 5$; $\mathcal{H}$ a handle decomposition of M with one (0)-handle and one (n)-handle, and $\mathcal{H}' = \{M_i', i=0,\ldots,n\}$ the dual handle decomposition. If $f \in T_{\mathcal{H}}$ then f is isotopic to $g \in T_{\mathcal{H}}$ such that the geometric intersection numbers of g equal the absolute value of algebraic intersections defined as follows

(a) for $3 \leq k \leq n-3$ on $\tilde{C}_k = H_k(\tilde{M}_k, \tilde{M}_{k-1}; Z) \cong \pi_k(M_k, M_{k-1}, x_o)$

(b) for $k = 2$, $n-2$ on $\pi_2(M_2, M_1, x_o)$ and $\pi_2(M_2', M_1', x_o')$

(c) for $k=1$, $n-1$ on $\pi_1(M_1, x_o)$ and $\pi_1(M_1', x_o')$.

The isotopy f_t, $0 \leq t \leq 1$, preserves $f_t(M_i) \subset \operatorname{int} M_i$, $i \geq 1$ and $f_t(x_o) = x_o$, $f_t(x_o') = x_o'$ so the algebraic intersection numbers are unchanged.

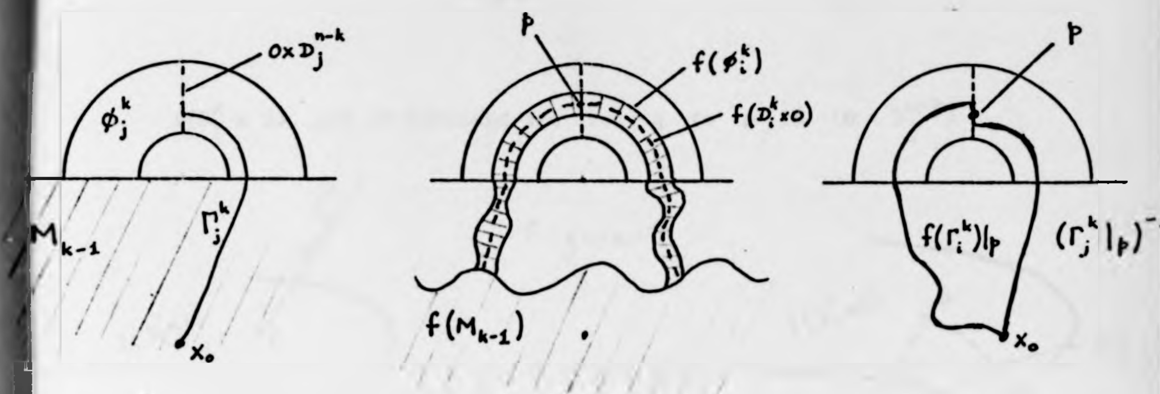
1.2 Remarks. If x_o is a base point chosen in the interior of the (0)-handle, and x_o' in the interior of the (n)-handle, we modify the definition of $f \in T_{\mathcal{H}}$ to include $f(x_o) = x_o$ and $f(x_o') = x_o'$. Then for $f \in T_{\mathcal{H}}$ the induced maps $f_{\#_i} : \pi_i(M_i, M_{i-1}, x_o) \circlearrowleft$ and $(f^{-1})_{\#_i} : \pi_i(M_i', M_{i-1}', x_o') \circlearrowleft$ $i \geq 1$, are defined.

Recall that the deck transformations of $\tilde{M}$ induce an action of $\pi_1(M, x_o)$ on $\tilde{C}_k$, $0 \leq k \leq n$, making it a free $Z[\pi_1]$-module with one generator for each (k)-handle of M. We construct a basis for $\tilde{C}_k$ as follows. First choose a base point x_o in the interior of the (0)-handle $\mathcal{O}^o$. Then we choose base paths Γ_i^k from x_o to each handle ϕ_i^k of M. This corresponds to choosing a particular lift of each core disc in $\tilde{M}$. Since M may not be orientable, we use the base paths Γ_i^k to orient the handles. Choose an orientation of the (0)-handle, ϕ^o, containing x_o. Since the Γ_i^k are embedded arcs, their tubular neighbourhoods in M are orientable submanifolds and inherit an orientation from ϕ^o. This induces an orientation $\mathrm{ORE}(\phi_i^k)$ for each handle. We then choose orientations $\sigma_{i,c}^k$ for the core discs $(D_i^k \times 0)$ and $\sigma_{i,t}^{n-k}$ for the transverse discs $(0 \times D_i^{n-k})$, so that $[\sigma_{i,c}^k, \sigma_{i,t}^{n-k}] = \mathrm{ORE}(\phi_i^k)$. With the chosen base paths and orientations, the core discs $(D_i^k \times 0)$ give a geometric basis for the $\tilde{C}_k$ as $Z[\pi_1]$-modules. We will also write these basis elements as ϕ_i^k.

Let $\tilde{x}_o$ be a base point of $\tilde{M}$ and $\tilde{f} : (\tilde{M}, \tilde{x}_o) \circlearrowleft$ the unique base point preserving lift of $f:(M, x_o) \circlearrowleft$, $f \in T_{\mathcal{V}}$. We define the algebraic intersection numbers $\tilde{a}_{ij}^k$ of f on $\tilde{C}_k$ to be the incidence numbers of $\tilde{f}_{*_k}$ in the chosen basis of $\tilde{C}_k$.

For $p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ the orientation number $\mathrm{ORE}(p) = \pm 1$ is defined by comparing $f(\sigma_{i,c}^k)$ at p with $\sigma_{j,c}^k$ we will write this as $\mathrm{ORE}(p) = < f(\sigma_{i,c}^k), \sigma_{j,c}^k >_p$. This comparison is well-defined because the projection $(\mathrm{proj})_j : f(D_i^k \times 0) \cap \phi_j^k \to (D_j^k \times 0)$ is locally a diffeomorphism near p. To determine the incidence numbers of $\tilde{f}_* : \tilde{C}_k \circlearrowleft$, we also associate an element $\alpha(p) \in \pi_1(M, x_o)$. For any $p \in \phi_j^k$, let $(\Gamma_j^k|_p)$ be the base path (Γ_j^k) extended to p by any path in ϕ_j^k. Then for $p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ define $\alpha(p)$ to be the class in $\pi_1(M, x_o)$ represented by the loop $(f(\Gamma_i^k)|_p)\,(\Gamma_j^k|_p)^{-1}$.

figure 3



ϕ_j^k and Γ_j^k $\qquad$ $p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ $\qquad$ $\alpha(p) \in \pi_1(M, x_o)$

Then
$$\tilde{a}_{ij}^k = \sum_{p \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})} \mathrm{ORE}(p)\, \alpha(p)$$

where the sum is defined in $Z[\pi_1]$. The matrix $\tilde{A}_k = (\tilde{a}_{ij}^k)$ represents the induced map $\tilde{f}_{*_k} : \tilde{C}_k \circlearrowleft$ in the chosen basis. (Following Shub and Sullivan's notation we write our matrix $\tilde{A}_k$ so that
$\tilde{f}_{*_k}(\phi_i^k) = \Sigma\, \tilde{a}_{ij}^k\, \phi_j^k$.)

Note that $\tilde{f}_{*_k}$ is not a $Z[\pi_1]$-module homomorphism, since
$\tilde{f}_{*_k}(\beta\, \phi_i^k) = f_*(\beta)\, \tilde{f}_{*_k}(\phi_i^k)$ where $\beta \in \pi_1(M, x_o)$ and $f_* : \pi_1(M, x_o) \circlearrowleft$.

We define an "absolute value" for $\tilde{a}_{ij}^k \in Z[\pi_1]$ by

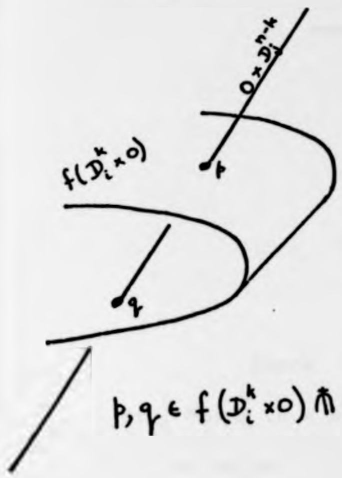
$$\Big| \sum_{\substack{\alpha \in \pi_1 \\ m_\alpha \in \mathbb{Z}}} m_\alpha \alpha \Big| = \sum_\alpha |m_\alpha| .$$

1.3 Proof of 1.1(a)

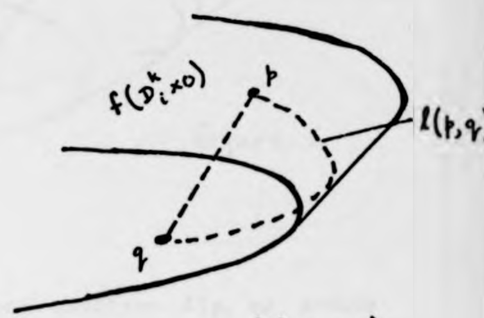
For $3 \leq k \leq n-3$ the cancelling lemma follows from a standard application of the Whitney lemma. It suffices to remove by isotopy all pairs $p, q \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$ whose contributions cancel in $\tilde{a}_{ij}^k$. This happens if and only if $\alpha(p) = \alpha(q)$ and $\mathrm{ORE}(p) = -\mathrm{ORE}(q)$. Construct a loop $\ell(p, q)$ with an embedded arc from p to q in

$f(D_i^k \times 0)$ and an embedded arc from q to p in $(0 \times D_j^{n-k})$

figure 4



$p, q \in f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k})$

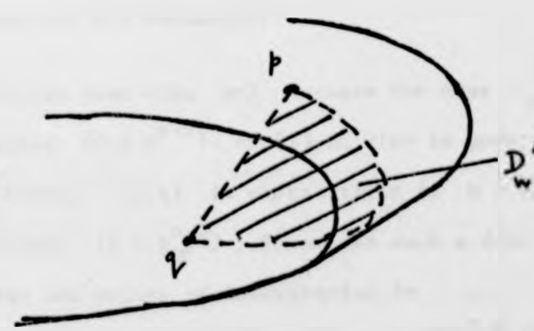


$\ell(p, q)$

It is easy to see that $\alpha(p) = \alpha(q)$ in $\pi_1(M, x_0)$ if and only if $\ell(p, q)$ is contractible in M. If p,q cancel in $\tilde{a}_{ij}^k$ then $\ell(p, q)$ is spanned by a (possibly singular) (2)-disc. If $\dim M \geq 5$ we can approximate this disc by an embedded (2)-disc D_w^2 spanning $\ell(p, q)$

figure 5

D^2_w spanning $\ell(p,q)$



Roughly speaking, we slide $f(D_i^k \times 0)$ back across D_w^2, opening up a a "notch" in $f(D_i^k \times 0)$, gaping around the transverse disc $(0 \times D_j^{n-k})$. (For details of the Whitney lemma see [10] or [16])

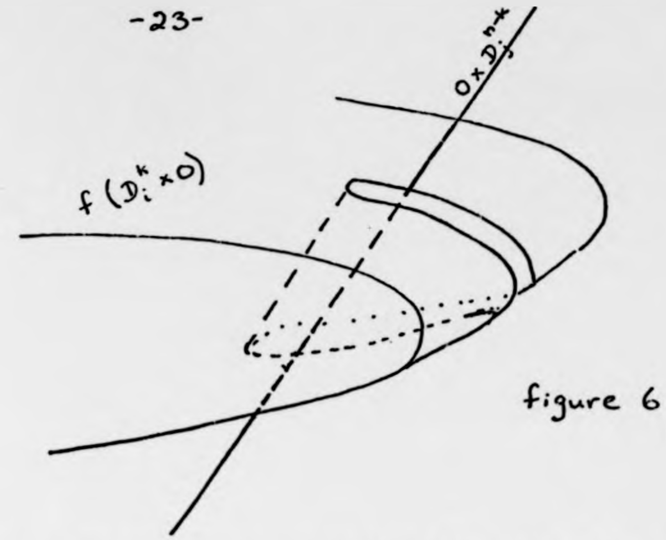


figure 6

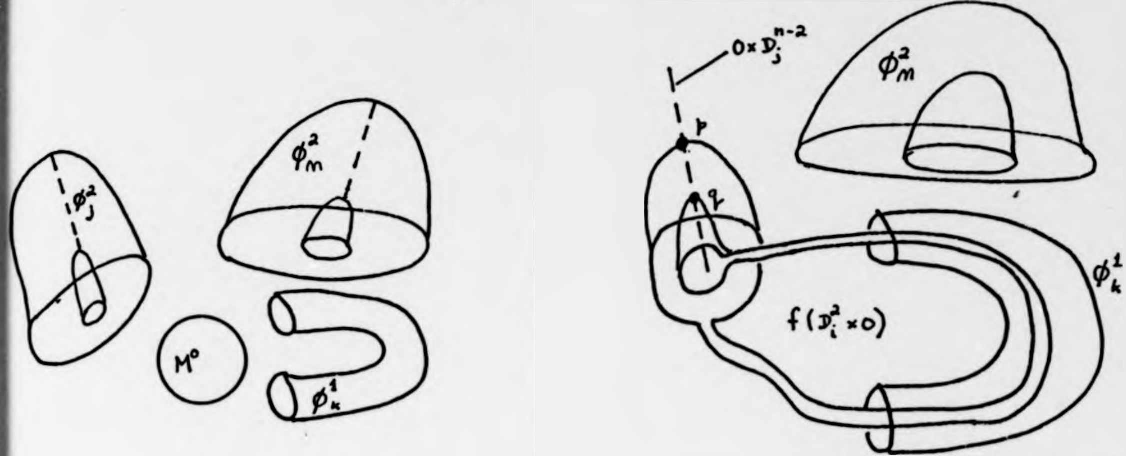
Since $1 < k < n-1$, by general position $\ell(p, q)$ avoids $f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k}) \setminus \{p, q\}$. Similarly, since $2 < k < n-2$, we can assume $\mathrm{int}\, D_w^2$ is disjoint from the discs $(0 \times D_j^{n-k})$ and $f(D_i^k \times 0)$. Therefore, by the Extension of Isotopy theorem [11], we can obtain an isotopy $f \simeq f_1$, which leaves f unchanged outside the inverse image of a small tubular neighbourhood of D_w^2. Therefore,

$$f_1(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k}) = f(D_i^k \times 0) \pitchfork (0 \times D_j^{n-k}) \setminus \{p, q\}$$

and all the other intersections are unchanged. $\square$

This argument breaks down when $k=2$ because the disc D_w^2 can meet the transverse discs $(0 \times D^{n-2})$ of (2)-handles in general position. In the example below, $\ell(p,q)$ is contractible in M, but any (2)-disc spanning it will meet $(0 \times D_m^{n-2})$. If we use such a disc in the Whitney lemma, then two new points of intersection in $f(D_i^2 \times 0) \pitchfork (0 \times D_m^{n-2})$ would be created for each point of $D_w^2 \pitchfork (0 \times D_m^{n-2})$.

figure 7



$\partial(\phi^2_m)$ kills $[\phi^1_k]$ in $\pi_1(M, x_0)$

$p, q \in f(D^2_i \times 0) \pitchfork (0 \times D^{n-2}_j)$. Problem: $\ell(p, q$ only contracts across ϕ^2_m (ϕ^2_j is represente by its transverse disc $(0 \times D^{n-2}_j)$).

In the simply connected case, Shub and Sullivan avoid this problem because they assume the handle decomposition has no (1) or (n-1) handles. Therefore, all the (2)-handles are trivially attached, and

$$(\mathrm{inc})_* : \pi_1(M_2 \setminus \bigcup_j (0 \times D^{n-2}_j)) \to \pi_1(M_2)$$

is injective. Similarly, in the proof of the s-cobordism theorem, we first eliminate the (1)-handles of the relative handle decomposition.

In the example above (figure 7), the problem is that $\tilde{f}_{*_2} : \tilde{C}_2 \circlearrowleft$ does not see that the loop $\ell(p,q)$ must pass through the (1)-handle ϕ^1_k because $[\phi^1_k] = 1$ in $\pi_1(M, x_0)$. This suggests we can avoid this problem if we calculate algebraic intersections on $\pi_2(M_2, M_1, x_0)$ which admits an action of the free group $\pi_1(M_1, x_0)$. We will show this is indeed the case, provided $\dim M \geq 5$.

The groups $\pi_i(M_i, M_{i-1}, x_o)$ were studied by Whitehead in the case of a CW complex [25] . Following Whitehead, we shall sometimes write ρ_i for $\pi_i(M_i, M_{i-1}, x_o)$ and ρ_1 for $\pi_1(M_1, x_o)$, and write $\pi_2(M_2, M_1, x_o)$ additively, although it is not in general abelian. The action of $\pi_1(M_1, X_o)$ can then be written multiplicatively. $\pi_2(M_2, M_1, x_o)$ is generated by elements of the form $x[\phi^2_i]$ where $x \in \pi_1(M_1, x_o)$ and $[\phi^2_i]$ is the element of $\pi_2(M_2, M_1, x_o)$ represented by the oriented, base-pathed core disc. There is a boundary map $d_2 : \pi_2(M_2, M_1, x_o) \to \pi_1(M_1, x_o)$ coming from the exact sequence of the pair (M_2 , M_1). This satisfies $d_2(x[\phi^2_i]) = x\, d_2[\phi^2_i]x^{-1}$ for $x \in \rho_1$. Clearly, $d_2(\rho_2)$ is a normal subgroup of ρ_1 , and $\bar{\rho}_1 \underset{\text{def.}}{=}$ $\rho_1/d_2(\rho_2) \cong \pi_1(M, x_o)$. For $x \in \rho_1$ we will write $\bar{x}$ for its class in $\bar{\rho}_1$.

The relations in $\pi_2(M_2, M_1, x_o)$ are of the form

$$A + B - A = d_2(A)\ (B) \tag{*}$$

where $A,B \in \pi_2(M_2, M_1, x_o)$, and these are the only relations. Notice it follows that $d_2^{-1}(1) \subseteq \text{centre}\ (\rho_2)$.

Note that in figure 7 above, the points $p,q \in f(D^2_i \times 0) \pitchfork (0 \times D^{n-2}_j)$ differ by the action of $[\phi^1_k] \in \pi_1(M_1, x_o)$ so they do not cancel in $f_{\#_2} : \pi_2(M_2, M_1, x_o) \circlearrowleft$.

Because of the relations (*), there is no canonical way to define the absolute value of algebraic intersections for $f_{\#_2}$. For example $\phi^2_j + \phi^2_i - \phi^2_j - \phi^2_i = d_2(\phi^2_j)\ (\phi^2_i) - \phi^2_i = \phi^2_j - d_2(\phi^2_i)\ (\phi^2_j)$.

Given a particular expression E_i of the form

$$f_{\#_2}([\phi^2_i]) = \epsilon_1 x_1\, \phi^2_{\ell_1} + \ldots\ldots + \epsilon_k x_k\, \phi^2_{\ell_k} \tag{**}$$

where $\epsilon_j = \pm 1$ and $x_j \in \pi_1(M_1, x_o)$, we can define $|E_{ij}| = \sum_{i_r = j} |\epsilon_r|$ i.e. we count the occurences of $[\phi_j^2]$ in E.

1.4 __Lemma.__ Let $\dim M \geq 5$, $\mathcal{H}$ a handle decomposition with one 0-handle and $f \in T_{\mathcal{H}}$. f is isotopic to $f_1 \in T_{\mathcal{H}}$ such that geometric intersections of f_1 agree with the absolute value of algebraic intersections calculated on $\pi_2(M_2, M_1, x_o)$; i.e. given a collection of expressions $E_i = f_{\#_2}[\phi_i^2]$, f is isotopic to $f_1 \in T_{\mathcal{H}}$ such that $g_{ij}^2(f_1) = |E_{ij}|$.

1.5 __Remark__ If the dimension of M is high enough (≥ 7), then 1.4 follows essentially by general position. Suppose $f_{\#_2}[\phi_i^2] = E$ where E is an expression of the form (**) in $\pi_2(M_2, M_1, x_o)$. Build a smooth model of the expression E, that is, an embedding $g : (D_i^2 \times 0), (S_i^1 \times 0) \to \text{int } M_2, \text{int } M_1$ which is fitted in the sense that it consists of complete core discs, and such that its geometric intersections with both (1) and (2)-handles are exactly as described in the expression E. (see (1.8) below.)

If we take a little care with the base paths, we obtain such a model of E homotopic to $f/(D_i^2 \times 0)$. Since $\pi_2(M_2, M_1, x_o) \cong \pi_2(\text{int } M_2, \text{int } M_1, x_o)$ there exists a homotopy

$$H : (D_i^2 \times 0) \times [0,1], (S_i^1 \times 0) \times [0, 1] \to \text{int } M_2, \text{int } M_1$$

such that $H^0 = f/(D_i^2 \times 0)$ and $H^1 = g$. Both H^0 and H^1 are smooth so we can approximate H by a smooth map G of the same pairs such that $G^0 = H^0 = f/(D_i^2 \times 0)$ and $G^1 = H^1 = g$. Then if $\dim M \geq 7$, by the Whitney Embedding theorem we can approximate G by an arbitrarily C^r close embedding, F. We choose F close enough so that F^0 is isotopic to G^0 and F^1 is isotopic to G^1. This yields an isotopy of $f/(D^2_1 \times 0)$ with the model g .

If $\dim M \geq 6$, by general position we can assume the track of the isotopy is disjoint from the images $f(D^k_m \times 0)$ for $k \leq 2$ and $m \neq 1$. By the extension of isotopy theorem we obtain an isotopy of f to $f_1 \in T_H$ such that $f_1/(D^2_1 \times 0) = g$ so $g^2_{1j}(f_1) = |E_{1j}|$ and all the other geometric intersection numbers are unchanged. This proves (1.4) when $\dim M \geq 7$. □

Using results of Haefliger, we can probably extend this argument to cover $\dim M = 6$.

Theorem (Haefliger [6]). Let M^n be a manifold of dimension n , V^r a closed manifold of dimension r and $f : V^r \to M^n$ a continuous map. If $2n > 3(r+1)$ and if $\pi_i(f) = 0$ for $i \leq 2r-n+2$, then two embeddings f_o and f_1 homotopic to f are isotopic.

$\pi_i(f) = 0$ means $f_{*_{i-1}} : \pi_{i-1}(V) \to \pi_{i-1}(M)$ is an isomorphism and $f_{*_i} : \pi_i(V) \to \pi_i(M)$ is surjective. If $n=5$ and $r=2$, we would need $f_* : \pi_1(V) \to \pi_1(M)$ surjective, but when $n=6$ only that $f_*: \pi_o(V) \to \pi_o(M)$ is surjective. Provided we can adapt Haefliger's proof for $\partial V \neq \emptyset$ (using $f_* : \pi_i(D^2_1\ S^1) \to \pi_i(M_2, M_1)$), we obtain a proof of (1.4) for $\dim M \geq 6$. In any case we will now prove it directly for $\dim M \geq 5$ using Whitehead's presentation of $\pi_2(M_2, M_1, x_o)$.

1.6 Proof of 1.4

We will show that the relations (*) in $\pi_2(M_2, M_1, x_o)$ $A+B-A = d_2(A)(B)$ represent precisely the changes in geometric intersection numbers which can be realized using the Whitney process. Suppose

$f_{\#_2}([\phi_i^2]) = E$ in $\pi_2(M_2, M_1, x_o)$ where E is some expression of the form (**)

$E = \epsilon_1\, x_1[\phi^2_{\ell_1}] + \ldots\ldots\ldots + \epsilon_k\, x_k\, [\phi^2_{\ell_k}]$, $\epsilon_\ell = \pm 1$, $x_\ell \in \pi_1(M_1, x_o)$.

We first deform $f/(D_i^2 \times 0)$ into a normal form without changing the geometric intersection numbers. It will follow that $f/(D_i^2 \times 0)$ is associated with a well defined expression F of the form (**), such that $f_{\#_2}([\phi_i^2]) = F$ and $g^2_{ij}(f) = |F_{ij}|$. Then if F transforms into F' by one application of the relations (*) it will follow, using the Whitney process, that $f/(D_i^2 \times 0)$ is isotopic to $f_1:(D_i^2 \times 0), (S_i^1 \times 0) \to$ $\to$ int M_2, int M_1 which is also in normal form and associated with F' . By the extension of isotopy theorem, we can extend f_1 to a diffeomorphism of M such that all the other geometric intersection numbers are unchanged. The lemma then follows by a finite induction.

First we need some definitions.

1.7 <u>Definition</u> (Shub-Sullivan) [16]. A handle decomposition is said to be <u>fitted</u> if the attaching maps $\phi_i^k : (S_i^{k-1} \times D_i^{n-k}) \to \partial M_{k-1}$ satisfy

(i) $\forall\, p \in D_i^{n-k}$, $\phi_i^k\,(S_i^{k-1} \times p)$ completely contains any core disc it meets in ∂M_{k-1} .

(ii) $\forall q \in S_i^{k-1}$, $\phi_i^k(q \times D_i^{n-k})$ is completely contained in any transverse disc it meets in ∂M_{k-1} . □

Shub and Sullivan prove that the attaching maps of a handle decomposition $\mathcal{H}$ can be isotoped to obtain a fitted decomposition, $\mathcal{H}_1$. If $f \in T_{\mathcal{H}}$ then f is isotopic to $f_1 \in T_{\mathcal{H}_1}$ such that $g^k_{ij}(f) = g^k_{ij}(f_1)$. Therefore, without loss of generality, we can assume that $\mathcal{H}$ is fitted and f is fitted with respect to $\mathcal{H}$ $(f \in F_{\mathcal{H}})$.

We also modify slightly the choice of base paths for the 2-handles. Since $\mathcal{H}$ is fitted and has a single 0-handle ϕ^0, the attaching maps of all handles must hit $\partial\phi^0$. We require that the base paths Γ_i^2 lie in $(M_1 \cup \phi_i^2)$ and enter (ϕ_i^2) only once and through some component of $(S_i^1 \times D_i^{n-2}) \cap \partial\phi^0$. Therefore, each Γ_i^2 is associated with an element $\gamma_i^2 \in \pi_1(M_1, x_0)$. We also choose base-points for each of the core discs $(D_i^2 \times 0)$, $z_0^i \in (S_i^1 \times 0) \cap \partial\phi^0$ lying in the component of $(S_i^1 \times D_i^{n-2}) \cap \partial\phi^0$ through which Γ_i^2 passes. By $[z_0^i]$ we will mean the component of $(S_i^1 \times D_i^{n-2}) \cap \partial\phi^0$ containing z_0^i.

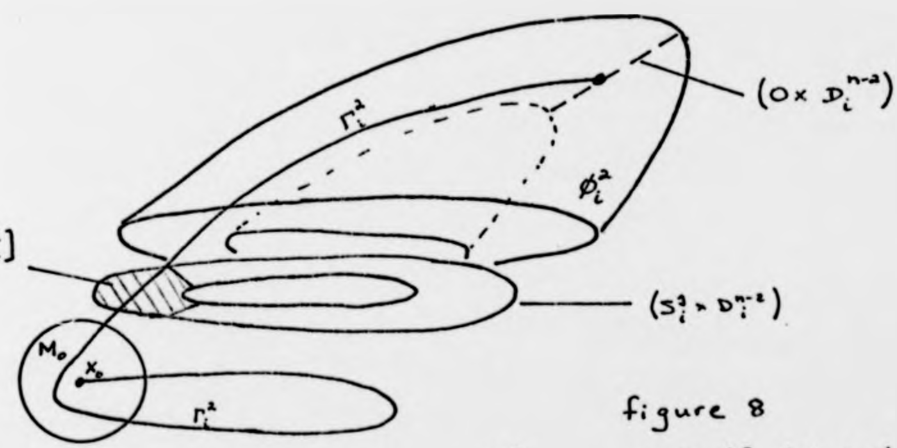


figure 8

Γ_i^2, and $[z_0^i]$ (shaded)

Since $f(M_0) \subset \text{int } M_0$, $f(z_0^i) \in \text{int } \phi^0$. By $[f(z_0^i)]$ we will mean the component of $f(D_i^2 \times 0) \cap \phi_0$ containing $f(z_0^i)$.

Finally, let $\pitchfork_{ij} = f(D_i^2 \times 0) \pitchfork (0 \times D_j^{n-2})$ and $\pitchfork_i = \bigcup_j \pitchfork_{ij}$.

1.8 <u>Definition.</u> Let $\mathcal{H}$ be a fitted handle decomposition and

$f : M \to M$ fitted on $(D_i^2 \times 0)$. Therefore $f(D_i^2 \times 0) \cap (\overline{M_2 - M_1})$ consists of complete core discs, one for each point $q \in \pitchfork_i$. If $q \in \pitchfork_{ij}$, by abuse of notation we will call this core disc $(D_j^2 \times q)$, and its boundary $(S_j^1 \times q)$.

We will say $f/(D_i^2 \times 0)$ is in normal form provided:

1.8.1 each point $p \in \pitchfork_i$ lies in a distinct component of $(f(D_i^2 \times 0) \setminus [f(z_o^i)])$ i.e. if $p,q \in \pitchfork_i$ then any path from p to q in $f(D_i^2 \times 0)$ meets $[f(z_o^i)]$

1.8.2 $(D_j^2 \times q)$ is connected to the rest of $f(D_i^2 \times 0)$ through the component of its base point $[z_o^j]$. That is, every path from $f(z_o^i)$ to q in $f(D_i^2 \times 0)$ passes through a component of $f(D_i^2 \times 0) \cap \emptyset^0$ bounded by $(S_j^1 \times q) \cap [z_o^j]$

1.8.3 every component of $f(D_i^2 \times 0) \cap \emptyset_m^1$ meets $f(S_i^1 \times 0)$, $\forall_m$.

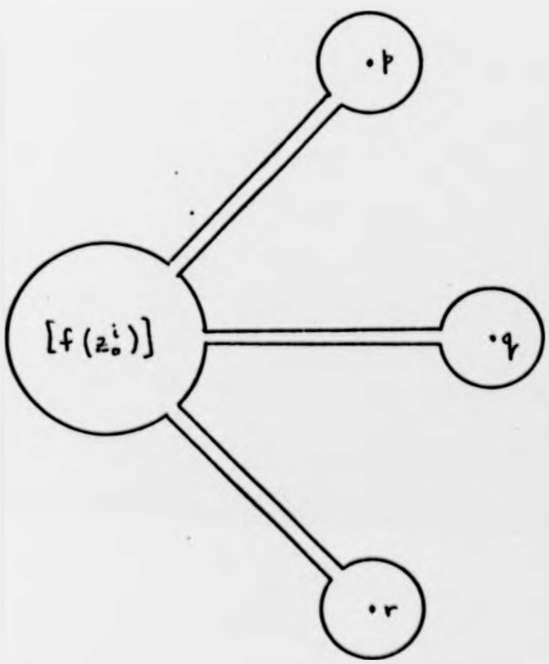


figure 9. $f(D_i^2 \times 0)$ in normal form. $p, q, r \in \pitchfork_i$

<u>Step 1</u> It is easy to see in the picture that when $f/(D_i^2 \times 0)$ is in normal form it is associated with a well defined expression of the form (**). For each $p \in f(D_i^2 \times 0) \pitchfork (0 \times D_j^{n-2})$ choose a path δ from $f(z_o^i)$ to p lying in $f(D_i^2 \times 0)$. By general position we can assume δ avoids $\pitchfork_i$ except at p. Truncating δ if necessary we obtain such a path which meets $[f(z_o^i)]$ in just one component. We then truncate δ at the other end to obtain a path from $f(z_o^i)$ to $[z_o^j]$ lying in the punctured disc $(f(D_i^2 \times 0) \setminus \pitchfork_i)$. Using radial projection $(\text{proj})\colon (M_2 \setminus \bigcup_{\ell} (0 \times D_\ell^{n-2})) \to \operatorname{int} M_1$ we obtain a path from $f(z_o^i)$ to $[z_o^j]$ lying in $f(D_i^2 \times 0) \cap M_1$. This path represents an element $\beta_p \in \pi_1(M_1, x_o)$. We claim β_p is independent of the choice of the path δ. Let δ' be another such path which meets $[f(z_o^i)]$ once. Now, $(\text{proj})_* : \pi_1(f(D_i^2 \times 0) \setminus \pitchfork_i, f(z_o^i)) \to \pi_1(M_1, x_o)$ is injective, so suppose $[\delta(\delta')^{-1}] \neq 1$ in $\pi_1(f(D_i^2 \times 0) \setminus \pitchfork_i, f(z_o^i))$. Then the loop $\delta(\delta')^{-1}$ must enclose some points $q \in \pitchfork_i$, $q \neq p$.

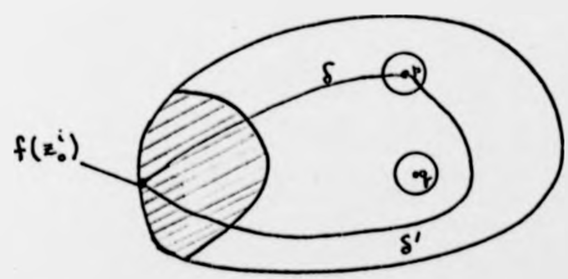


figure 10

$p, q \in \pitchfork_{ij}$, $[f(z_o^i)]$ shaded

$\delta(\delta')^{-1}$ encloses q

The shaded region is $[f(z_o^i)]$.

By 1.8.1 p and q are separated in $f(D_i^2 \times 0)$ by $[f(z_o^i)]$. Therefore we must be in one of the following situations:

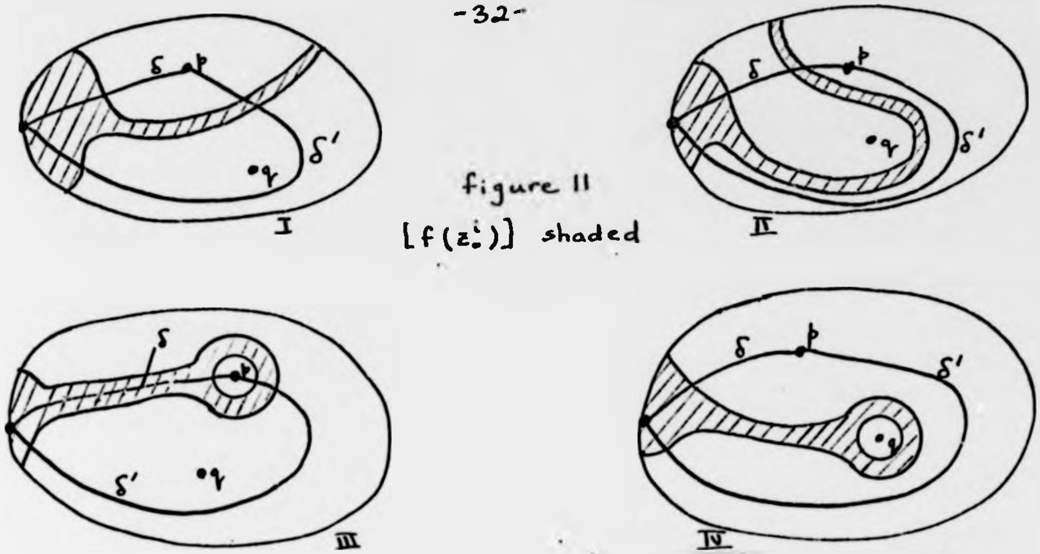


figure 11
$[f(z_o^i)]$ shaded

In cases I, II, and III, one of the paths re-enters $[f(z_o^i)]$. In case IV it follows from 1.8.3 that the 2-handle ϕ_j^2 that q lies in must be trivially attached and hence $d_2[\phi_j^2] = 1$ and $[\delta(\delta')^{-1}] = 1$. Therefore, $\beta_p \in \pi_1(M_1, x_o)$ is well-defined. The orientation numbers $ORE(p) = \pm 1$ are defined as usual.

Since $\pi_2(M_2, M_1, x_o)$ is not in general abelian, we also need a natural ordering of $\pitchfork_i$, which we describe as follows. For each $p \in \pitchfork_i$ let δ_p be a path representing β_p as above. By 1.8.1 the δ_p are not able to meet outside $[f(z_o^i)]$. Therefore, we can find a family of such paths which meet only in the single point $f(z_o^i)$. Choose a small $\varepsilon > 0$ such that $B_\varepsilon(f(z_o^i)) \cap f(D_i^2 \times 0) \subset [f(z_o^i)]$ and make all the δ_p transverse to its boundary $S_\varepsilon^1(f(z_o^i)) \cap f(D_i^2 \times 0)$.

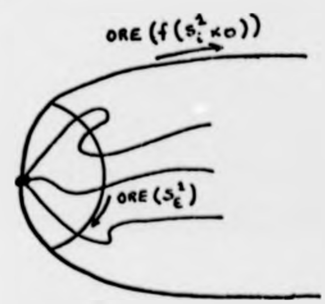


figure 12

Some of the paths δ_p may hit $S^1_\varepsilon(f(z^i_o))$ several times before leaving $[f(z^i_o)]$, but this will not affect the order. Let the orientation of $f(S^1_i \times 0)$ induce an orientation on $S^1_\varepsilon(f(z^i_o))$ and enumerate the points $p \in \mathfrak{m}_i$ according to the ordering of $\delta_p \cap S^1_\varepsilon(f(z^i_o))$.

This ordering is independent of the choice of the family of paths δ_p, up to the position of $q \in \mathfrak{m}_{ij}$ where ϕ^2_j is trivially attached onto $\partial\phi^o$. For, suppose δ_p, δ_q and δ'_p, δ'_q are paths in two such families which reverse the relation of p and q. Then it is easy to see that either p is inside the closed curve $\delta_q(\delta'_q)^{-1}$ or else q is inside the closed curve $\delta_p(\delta'_p)^{-1}$.

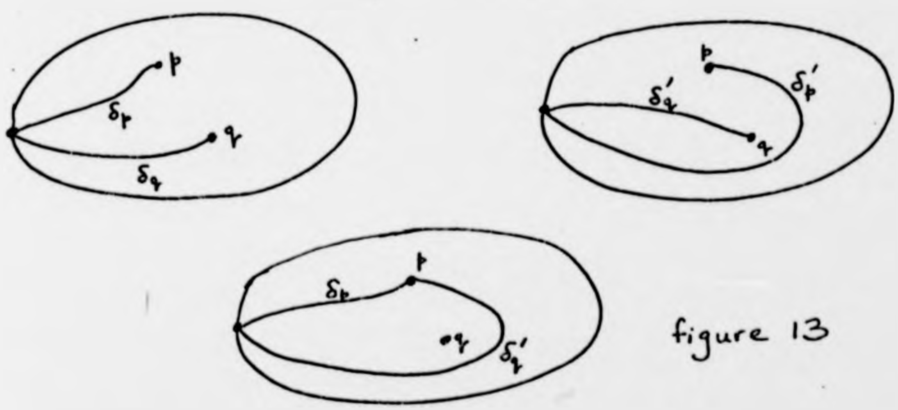


figure 13

As before, this can only happen when either p or q lies in a trivially attached 2-handle ϕ^2_j. This ambiguity will not affect our argument, however, since $d_2^{-1}(1) \subset$ centre $(\pi_2(M_2, M_1, x_o))$.

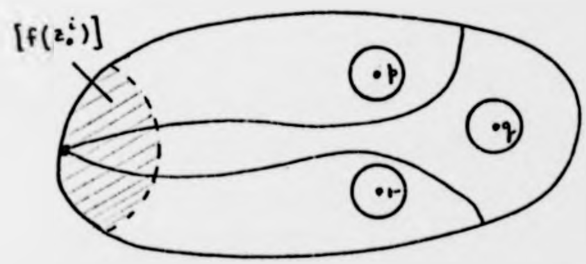
Given $f/(D^2_i \times 0)$ in normal form, we associate the expression F as follows. Recall that γ_i is the element of $\pi_1(M_1, x_o)$ associated with the base path Γ^2_i. For $p \in \mathfrak{m}_{ij}$ let $\beta'_p = f_{\#_1}(\gamma_i)\,\beta_p(\gamma_j)^{-1}$, and suppose the points of $\mathfrak{m}_i$ are enumerated as above $p_1, p_2, \ldots, p_k$. Then we define F to be the expression:

$$F = \mathrm{ORE}(p_1)\, \beta'_{p_1}[\emptyset^2_{j_1}] + \ldots\ldots\ldots\ldots + \mathrm{ORE}(p_k)\, \beta'_{p_k}[\emptyset^2_{j_k}] \quad .$$

Clearly, $f_{\#_2}[\emptyset^2_i] = F$ in $\pi_2(M_2, M_1, x_o)$ and $\beta^2_{ij}(f) = |F_{ij}|$.

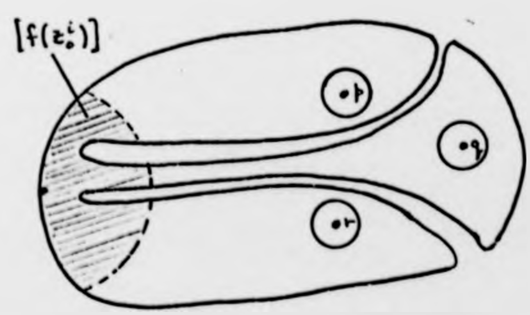
Step 2 We now show how $f/(D^2_i \times 0)$ can be isotoped to be in normal form. Since f is already fitted, $f(D^2_i \times 0) \cap \overline{(M_2 - M_1)}$ consists of a finite number of core discs of 2-handles, one for each point in $\pitchfork_i$. Therefore $f(D^2_i \times 0) \cap M_1$ is connected. Choose disjoint arcs from $f(S^1_i \times 0)$ to $f(z^i_o)$ which separate all the components of $f(D^2_i \times 0) \cap \overline{(M_2 - M_1)}$.

figure 14. $f(D^2_i \times 0)$ showing arcs separating $p, q, r \in \pitchfork_i$



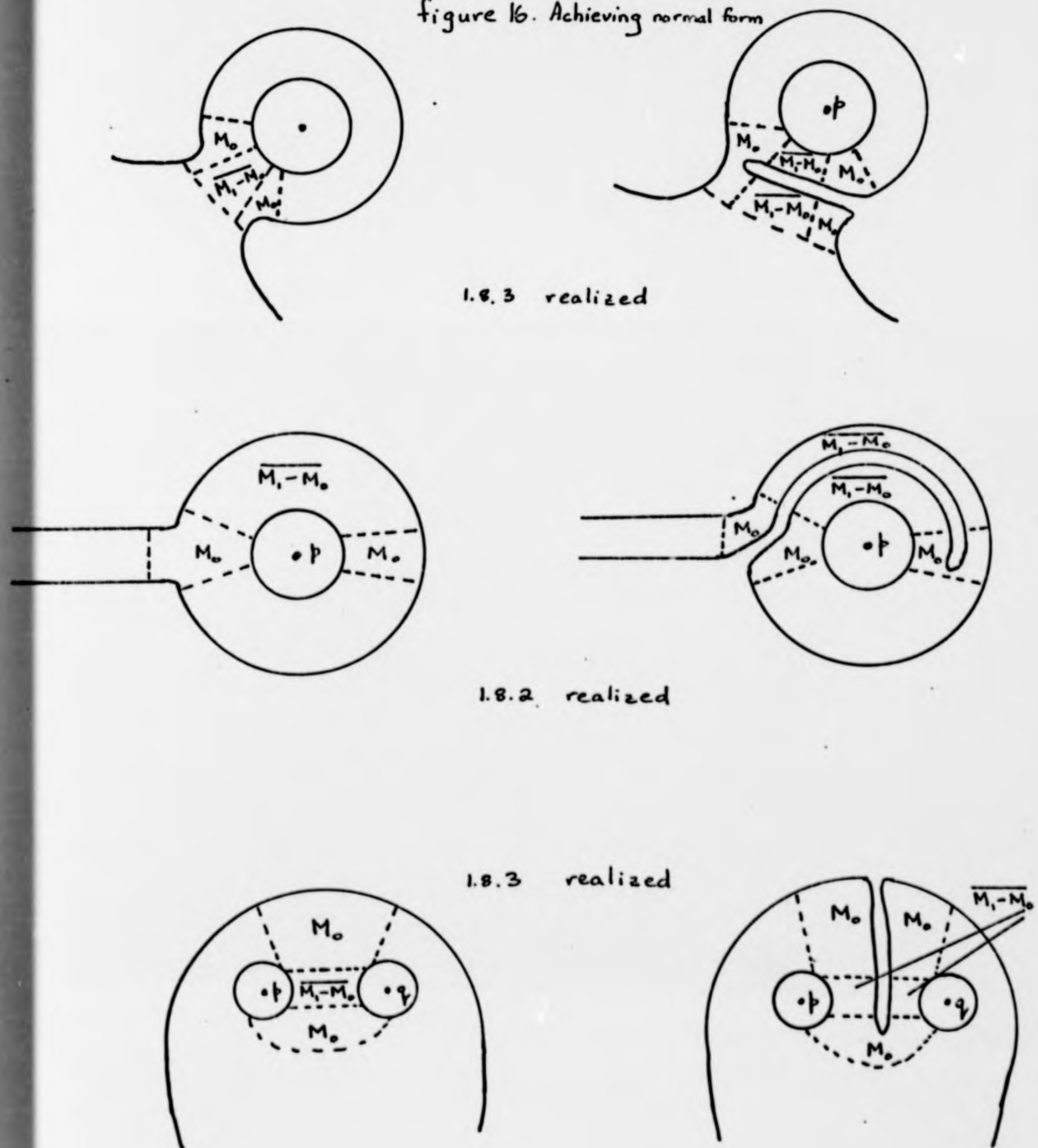
Then "unzip" $f(S^1_i \times 0)$ along these arcs as far as $[f(z^i_o)]$

figure 15
$f(D^2_i \times 0)$ "unzipped"



That is, we first define such an isotopy of $f/(S^1_i \times 0)$ and then extend it to an isotopy of $f/(D^2_i \times 0)$. This guarantees 1.8.1 is satisfied. Continuing the same process we can easily realize 1.8.2 and 1.8.3 as in the following diagrams.

figure 16. Achieving normal form



Notice that these steps yield an isotopy $f_t \quad 0 \leq t \leq 1$ of $f/(D_i^2 \times 0)$ which satisfies $f_t(D_i^2 \times 0) \subset f_o(D_i^2 \times 0)$ and $f_t(S_i^1 \times 0) \subset \text{int } M_1$, $\forall t \in [0, 1]$. By the extension of isotopy theorem, we obtain an isotopy defined on all of M, $f \underset{\text{iso}}{\simeq} f_1$ such that $f_1 \in T_{\mathcal{U}}$, $f_1/(D_i^2 \times 0)$ is in normal form, and the images of all the other central core discs $f_1(D_\ell^k \times 0)$ are unchanged, $\ell \neq i$

Step 3 Finally, we show that when $f/(D_i^2 \times 0)$ is in normal form, the Whitney process realizes the relations (*) . Let $f/(D_i^2 \times 0)$ be in normal form associated with an expression F and suppose $F \Rightarrow F_1$ by one use of the relations (*) . For simplicity suppose

$$F = x[\emptyset_j^2] + y[\emptyset_k^2] - x[\emptyset_j^2]$$

and

$$F_1 = xd_2[\emptyset_j^2]x^{-1}y\,[\emptyset_k^2] \; .$$

Let $p, q \in f(D_i^2 \times 0) \pitchfork (0 \times D_j^{n-2})$ correspond to the terms $\pm x\,[\emptyset_j^2]$. Construct a loop $\ell(p, q)$ using the paths δ_p, δ_q and an arc in $(0 \times D_j^{n-2})$. Since $(x\,x^{-1})$ contracts in M_1 , $\ell(p, q)$ is spanned by a disc in $M_1 \cup \emptyset_j^2$.

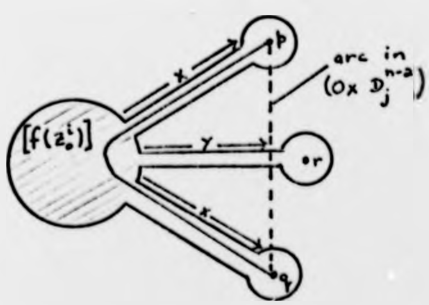
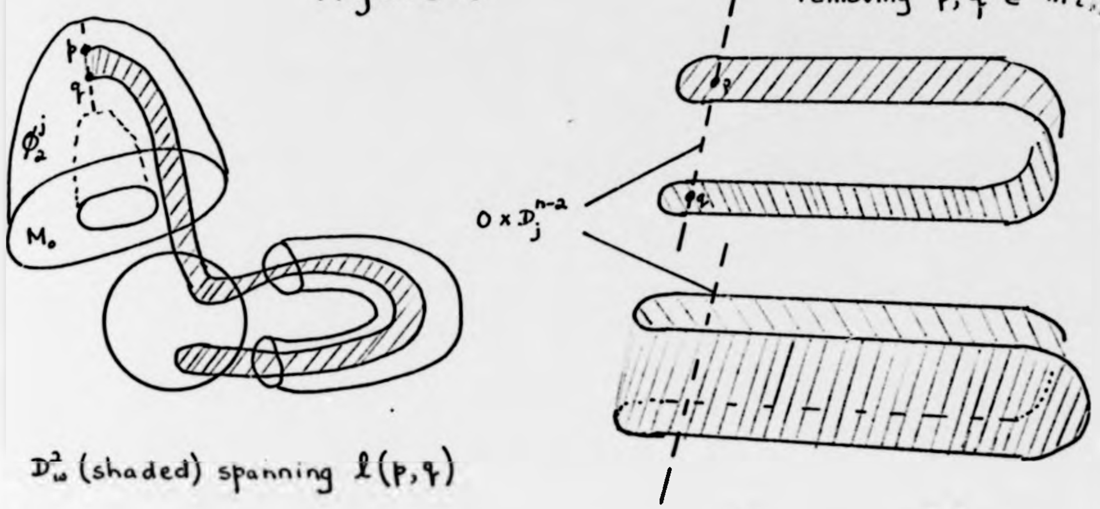


figure 17
$\ell(p, q) = 1$ in $\pi_1(M_1, x_0)$

Since δ_p and δ_q enter $\emptyset_j^2$ through the same component of $(\emptyset_j^2 \cap \partial\emptyset^o)$, we can choose a disc D_w^2 spanning $\ell(p, q)$ which meets $(0 \times D_j^{n-2})$ in the chosen arc form p to q .

figure 18



D_w^2 (shaded) spanning $\ell(p,q)$

Therefore no new intersections in $f(D_i^2 \times 0) \pitchfork (0 \times D_m^{n-2})$, any m, will be created when we perform the Whitney process.

In our example $F \Rightarrow F_1$ let $r \in f(D_i^2 \times 0) \pitchfork (0 \times D_k^{n-2})$ correspond to the middle term $y[\phi_k^2]$ in F . Notice that any arc from $f(z_o^i)$ to (r) representing β_r will cross the loop $\ell(p, q)$. Therefore when we perform the Whitney process the element $xd_2([\phi_j^2])x^{-1}$ is introduced into $(\beta_r)^{\cdot}$ as in the following picture.

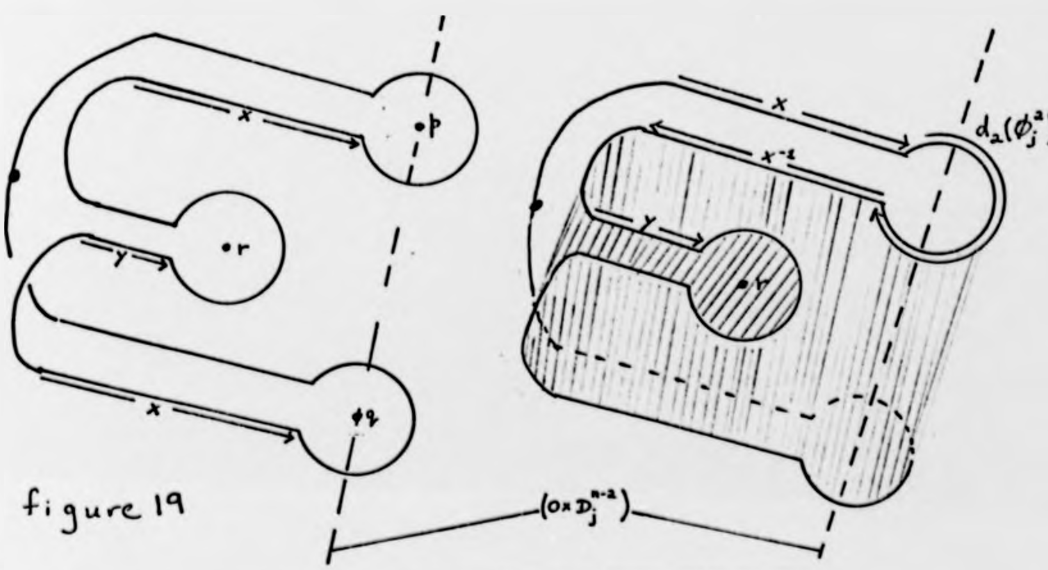


figure 19

This completes the proof of 1.4. □

1.9 It only remains to show that in the case $k=1$ we can realize the absolute value of algebraic intersections defined on the free group $\pi_1(M_1, x_o)$. Since $\mathcal{H}$ has a single 0-handle, $\pi_1(M_1, x_o)$ is the free group on the 1-handles. Let $[\emptyset_k^1]$ be the element represented by the oriented core disc $(D_k^1 \times 0)$. The "absolute value" $|\tilde{a}_{ij}^1|$ is defined by counting the occurrences of $[\emptyset_j^1]^{\pm 1}$ in the reduced word for $f_{\#_1}([\emptyset_i^1])$.

Provided $\dim M \geq 4$, no linking of the $f(D_k^1 \times 0)$ can occur. Therefore, we can with draw any homotopically trivial loops in $f(D_k^1 \times 0)$ by an isotopy. Since $f \in T_{\mathcal{H}}$, after finitely many such operations we have $g_{ij} = |\hat{a}_{ij}^1|$. □

CHAPTER TWO

CHAIN HOMOTOPY

In view of the previous chapter, given a handle decomposition $\mathcal{H}$ and a diffeomorphism $f: M \to M$, we need to know which chain maps $E: \rho(M,\mathcal{H})\circlearrowleft$ are realized by diffeomorphisms $g \in T_{\mathcal{H}}$ isotopic to f. If $f \in T_{\mathcal{H}}$ then E must be chain homotopic to $f_\#: \rho(M,\mathcal{H})\circlearrowleft$. We will show that this suffices, provided that E_n fulfils a natural condition satisfied by the induced maps of cellular homeomorphisms. The first sections of this chapter are devoted to algebraic preliminaries, giving the appropriate definition of chain homotopy for $\rho(M,\mathcal{H})$. All this material is due to Whitehead [25]. In the second half of the chapter we prove our theorem realizing such chain homotopies $f_\# \simeq E$ by an isotopy of f.

2.1) We first describe Whitehead's definition of an abstract homotopy system (ρ_k, d_k). Let ρ_1 be a free group, and ρ_2 an additive but not necessarily abelian group which admits an action of ρ_1 written as multiplication. Let $d_2: \rho_2 \to \rho_1$ be a homomorphism such that $d_2(xa) = xd_2(a)x^{-1}$ for $x \in \rho_1$, $a \in \rho_2$. Suppose $\{a_i \mid i \in I\}$ is a collection of elements of ρ_2 such that the elements xa_i, $x \in \rho_1$, $i \in I$, generate ρ_2. ρ_2 is a <u>free crossed (ρ_1, d_2) - module</u> if the only relations among these generators are of the form:

$$xa_i + ya_j - xa_i = (xd_2(a_i)x^{-1}y)\,a_j \qquad (*)$$

and if for all $x,y \in \rho_1$ and a_i, $i \in I$ $xa_i = ya_i$ if and only if equality follows from the relations $(*)$. We call the elements a_i, $i \in I$ a <u>basis</u> for ρ_2. If $\mathcal{H}$ is a handle decomposition of M, then $\pi_2(M_2, M_1, x_0)$ is a free crossed $(\pi_1(M_1, x_0), d_2)$ - module with basis

$\{ [\varphi_1^2], \ldots, [\varphi_{r_2}^2] \}$.

Let ρ_2 be a free crossed (ρ_1, d_2) - module. Recall that $\bar{\rho}_1$ is the factor group $\rho_1 / d_2(\rho_2)$ and $\bar{x}$ is the coset of $x \in \rho_1$. If $\rho_2' \xrightarrow{d_2'} \rho_1'$ is another free crossed module and $f_1 : \rho_1 \to \rho_1'$ is a homomorphism such that $f_1(d_2(\rho_2)) \subset d_2'(\rho_2')$ then we will write the induced homomorphism as $\bar{f}_1 : \bar{\rho}_1 \to \bar{\rho}_1'$.

Notice that if $a \in \rho_2$ and $d_2(a) = 1$, then for any $b \in \rho_2$ $\quad a + b - a = d_2(a)b = b$ so $d_2^{-1}(1) \subset \text{centre}(\rho_2)$. Therefore, if $a \in d_2^{-1}(1)$ and $x, y \in \rho_1$ satisfy $\bar{x} = \bar{y}$, then $xa = ya$ in ρ_2, i.e. $d_2^{-1}(1)$ admits an action of $\bar{\rho}_1$.

A <u>homotopy system</u> $\rho = (\rho_k, d_k)$ is a sequence of groups and homomorphisms

$$\rho_n \xrightarrow{d_n} \rho_{n-1} \xrightarrow{d_{n-1}} \cdots \rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1 \to 1$$

such that ρ_1 is a free group, ρ_2 is a free crossed (ρ_1, d_2) - module, and ρ_k is a free $Z[\bar{\rho}_1]$ - module for all $k \geq 3$. The boundary maps d_k $k \geq 3$ preserve the action of $\bar{\rho}_1$, i.e. if $a \in \rho_k$ and $\bar{x} \in \bar{\rho}_1$ then $d_k(\bar{x}a) = \bar{x}d_k(a)$. Of course, $d_k d_{k+1} = 0$ for $k \geq 3$ and $d_2 d_3 = 1$. We also assume that each group ρ_k has a preferred basis

$\{ [a_1^k] \ldots [a_{r_k}^k] \}$.

We now define homomorphisms of homotopy systems. Let G, G' be groups which admit actions of groups γ, γ', written multiplicatively, and let $E: G \to G'$ and $\varphi: \gamma \to \gamma'$ be group homomorphisms. We will say E is an operator homomorphism associated with φ if for all $x \in \gamma$ and $a \in G$, $E(xa) = \varphi(x) E(a)$.

Let ρ, ρ' be two homotopy systems. A homomorphism $E: \rho \to \rho'$ is a family $E_i: \rho_i \to \rho'_i$ satisfying $E_{i-1} d_i = d'_i E_i$ such that $E_2: \rho_2 \to \rho'_2$ is an operator homomorphism associated with $E_1: \rho_1 \to \rho'_1$ and for $k \geq 3$ $E_k: \rho_k \to \rho'_k$ is an operator homomorphism associated with $\bar{E}_1: \bar{\rho}_1 \to \bar{\rho}'_1$.

We will also need the notion of a crossed homomorphism. Let G be an additive group which admits the action of a multiplicative group γ. A map $h: \gamma \to G$ is said to be a crossed homomorphism if for all $x, y \in \gamma$ $h(xy) = h(x) + xh(y)$. Let γ' be another multiplicative group and $\varphi: \gamma' \to \gamma$ a homomorphism. A map $h: \gamma' \to G$ is said to be a crossed homomorphism associated with φ if for all $x, y \in \gamma'$ $h(xy) = h(x) + \varphi(x)h(y)$.

2.2) Let $\rho = (\rho_k, d_k)$ and $\rho' = (\rho'_k, d'_k)$ be homotopy systems and $E, E': \rho \to \rho'$ homomorphisms as defined above. A chain homotopy of E to E' consists of an element $\omega \in \rho'_1$ and a family of maps $\xi_k: \rho_{k-1} \to \rho'_k$, where $\xi_2: \rho_1 \to \rho'_2$ is a crossed homomorphism associated with $E_1: \rho_1 \to \rho'_1$ and for $k \geq 3$ $\xi_k: \rho_{k-1} \to \rho'_k$ is an operator homomorphism associated with $E_1: \rho_1 \to \rho'_1$. We require that for:

$$k \geq 3 \quad \bar{\omega} E'_k - E_k = d'_{k+1} \xi_{k+1} + \xi_k d_k$$

$$k = 2 \quad \omega E'_2 - E_2 = d'_3 \xi_3 + \xi_2 d_2 ,$$

and for all $x \in \rho_1$

$$\omega E'_1(x) \omega^{-1}(E_1(x))^{-1} = d'_2 \xi_2(x) .$$

If this is satisfied we will write $E \underset{\xi,\omega}{\simeq} E'$.

Note: Since ρ'_1 is not in general abelian, the map $x \to E'_1(x) (E_1(x))^{-1}$ is not in general a homomorphism. This accounts for the special form of $\xi_2 : \rho_1 \to \rho'_2$.

We state the following standard result adapted to the present definitions:

<u>Theorem</u> (Whitehead [25 Theorem 5]). Let $f, g \in T_{\mathfrak{A}}$ and suppose f is homotopic to g. Then $f_* \simeq g_*$ on $\rho(M, \mathfrak{A})$. ▫

2.3) To each homotopy system $\rho = (\rho_k, d_k)$ we associate a chain complex $C(\rho) = (C_k, \partial_k)$. Let C_k be a free module over the group ring $Z[\bar{\rho}_1]$ with a basis $\{ a_1^k, \ldots, a_{r_k}^k \}$ in 1-1 correspondence with the chosen basis $\{ [a_1^k], \ldots, [a_{r_k}^k] \}$ of ρ_k. The chain groups C_k are related to the groups ρ_k by projections $h_k : \rho_k \to C_k$, each a map of the appropriate algebraic type induced by the 1-1 correspondence of bases. Let C_0 be a free $Z[\bar{\rho}_1]$ - module with the single generator a^0. For $k \geq 3$, $h_k : \rho_k \to C_k$ is a $Z[\bar{\rho}_1]$ - module isomorphism. $h_2 : \rho_2 \to C_2$ is defined by $h_2(x[a_i^2]) = \bar{x} a_i^2$.

It follows that h_2 is surjective with kernel the commutator subgroup of ρ_2 i.e. C_2 is ρ_2 abelianized. Finally $h_1 \colon \rho_1 \to C_1$ is the unique crossed homomorphism induced by the 1-1 correspondence of bases and associated with the quotient map $\rho_1 \to \bar{\rho}_1$.

For $k \geq 2$, the boundary maps $\partial_k \colon C_k \to C_{k-1}$ are defined to be the unique $Z[\bar{\rho}_1]$ - module homomorphisms such that $h_{k-1} d_k = \partial_k h_k$. $\partial_1 \colon C_1 \to C_0$ is defined by $\partial_1(a_i^1) = ([\overline{a_i^1}] - 1)\, a^0$. We obtain the following commutative diagram:

$$
\begin{tikzcd}
\rho_n \arrow[r, "d_n"] \arrow[d, "h_n"'] & \rho_{n-1} \arrow[r] \arrow[d, "h_{n-1}"'] & \cdots\cdots \arrow[r] & \rho_3 \arrow[r, "d_3"] \arrow[d, "h_3"'] & \rho_2 \arrow[r, "d_2"] \arrow[d, "h_2"'] & \rho_1 \arrow[d, "h_1"'] & \\
C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0
\end{tikzcd}
$$

Let $h \colon \rho \to C$ and $h' \colon \rho' \to C'$ be two homotopy systems. A chain map $F \colon C \to C'$ will be a family of operator homomorphisms associated with a homomorphism $\theta \colon \bar{\rho}_1 \to \bar{\rho}'_1$ such that $F_{k-1}\, \partial_k = \partial'_k F_k$, $F_k(\bar{x} a_i^k) = \theta(\bar{x}) F_k(a_i^k)$, and $F_0(a^0) = (a^0)'$. If there is a homomorphism $E \colon \rho \to \rho'$ such that $h'_k E_k = F_k h_k$ we will say, depending on the point of view, that F is induced by E and that E is a lift of F.

2.4) <u>Theorem</u> (Whitehead [25 Theorem 9]. Let $h \colon \rho \to C$ and $h' \colon \rho' \to C'$ be homotopy systems. A homomorphism $E \colon \rho \to \rho'$ induces a unique chain map which we will write as

$E_*: \mathcal{C} \to \mathcal{C}'$. A chain map $F: \mathcal{C} \to \mathcal{C}'$ has at least one lift, which is uniquely determined by the lift of $F_1: \mathcal{C}_1 \to \mathcal{C}_1'$. Any two lifts of the same chain map are chain homotopic upstairs. □

Let $h: \rho \to \mathcal{C}$ and $h': \rho' \to \mathcal{C}'$ be homotopy systems and $F, F': \mathcal{C} \to \mathcal{C}'$ chain maps associated with homomorphisms $\theta, \theta': \bar{\rho}_1 \to \bar{\rho}_1'$. F is chain homotopic to F' if there is an element $\bar{\omega} \in \bar{\rho}_1'$ and a family of operator homomorphisms $\eta_k: C_{k-1} \to C_k'$ associated with θ such that $\bar{\omega} F_k' - F_k = \partial_{k+1}' \eta_{k+1} + \eta_k \partial_k$ for all $k \geq 0$ ($\eta_0 \partial_0 = 0$).

Let $E, E': \rho \to \rho'$ be homomorphisms and (ξ, ω) a chain homotopy of E to E'. A chain homotopy $(\xi_*, \bar{\omega})$ is induced downstairs on $\mathcal{C} \to \mathcal{C}'$, by $\xi_{*k} h_{k-1} = h_k' \xi_k$ for $k \geq 2$ and $\xi_{*1}(a^0) = h_1'(\omega)$. Therefore, if $E \simeq E'$ then $E_* \simeq (E')_*$. The converse is also true.

2.5) <u>Theorem</u> (Whitehead [25 Theorem 10]). Let E, E' be as above; then $E \underset{\xi,\omega}{\simeq} E'$ if and only if $E_* \underset{\xi_*,\bar{\omega}}{\simeq} (E')_*$. □

By a chain equivalence $E: \rho \to \rho'$ or $F: \mathcal{C}(\rho) \to \mathcal{C}(\rho')$ we mean in each case a morphism which has a chain homotopy inverse.

2.6) <u>Corollary</u> (Whitehead [25 Theorem 12]). Let $E: \rho \to \rho'$ be a homomorphism. E is a chain equivalence if and only if E_* is one. □

This completes the algebraic preliminaries.

The main result of this chapter is the following lemma:

2.7) Chain Homotopy Lemma. Let M be a connected closed manifold, dimension $(M) \geq 4$, $\mathcal{H}$ a fitted handle decomposition with a single (0)-handle and a single (n)-handle. Let $f \in T_{\mathcal{H}}$, $f_{\#}: \rho(M, \mathcal{H}) \circlearrowleft$, and $E: \rho(M, \mathcal{H}) \circlearrowleft$ be a homomorphism such that $f_{\#} \underset{(\xi, \omega)}{\simeq} E$. Suppose further that the matrix representing E_n satisfies the condition that its single entry is of the form $\pm \gamma$ where $\gamma \in \pi_1(M, x_0)$. (We will call homomorphisms E satisfying this condition 'monic'.) Then f is isotopic to $g \in T_{\mathcal{H}}$ such that $g_{\#} = E$.

2.8) We first prove (2.7) when $\xi_n = 0$, without the restriction on E_n.

Case 1 $3 \leq k \leq n-1$. Let $\xi_k([\varphi_i^{k-1}]) = \sum_{j=1}^{r_k} \xi_{ij}^k [\varphi_j^k]$

where $\xi_{ij}^k \in Z[\pi_1]$. We show how to realize a single term of the form $\xi_{ij}^k = \epsilon\alpha$, $\epsilon = \pm 1$, $\alpha \in \pi_1(M, x_0)$, and the result follows by finite induction. Choose an embedded arc l_α from $p \in f(D_i^{k-1} \times 0)$ to $q \in (0 \times D_j^{n-k})$ such that

$[f(\Gamma_i^{k-1})|_p \, l_\alpha (\Gamma_j^k|_q)^{-1}] = \alpha$ in $\pi_1(M, x_0)$.

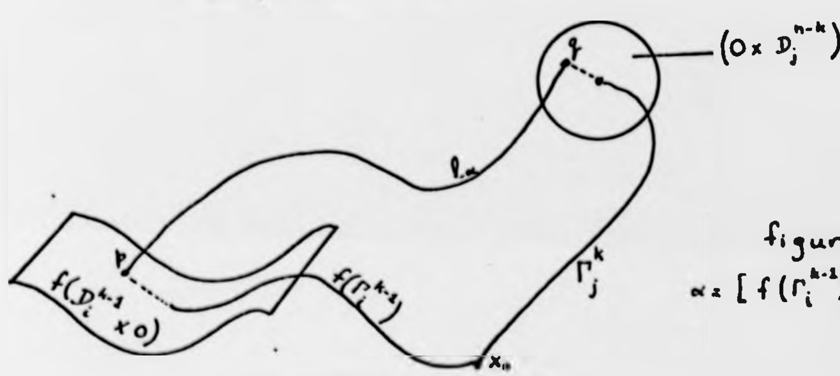


figure 20

$\alpha = [f(\Gamma_i^{k-1})|_p \, l_\alpha (\Gamma_j^k|_q)^{-1}]$

We choose p to lie in $f(\operatorname{int}(D_i^{k-1} \times 0)) \setminus \hbar_i$.

Since $k > 1$, by general position we can make l_α avoid the transverse discs of (k)-handles except at q; then by radial projection in the k-core directions we obtain l_α lying in $\operatorname{int}(M_{k-1} \cup \varphi_j^k)$. Since $k < n$, by general position we can ensure that l_α avoids the images $f(D_m^{k-1} \times 0)$, $m \neq i$, and meets $f(D_i^{k-1} \times 0)$ only in the point p (this is where the proof breaks down when $k = n$). Similarly, we can assume l_α avoids the images of all the central core discs of dimension $<(k-1)$

To create the linking number $\xi_{ij}^k = \varepsilon\alpha$, first extend l_α a little past $(0 \times D_j^{n-k})$. Since l_α is an embedded arc, it has a tubular neighbourhood $N_{\varepsilon_1}(l_\alpha)$ which is diffeomorphic to $D^1 \times D^{n-1}$. Pick ε, $0 < \varepsilon < \varepsilon_1$, small enough so that the ball $B_\varepsilon(p) \cap f(D_i^{k-1} \times 0)$ is a small $k-1$ disc $f(D_\varepsilon^{k-1}) \subset f(\operatorname{int}(D_i^{k-1} \times 0)) \setminus \hbar_i$. We still have $N_\varepsilon(l_\alpha) \cong D^1 \times D^{n-1}$. In this tube we slide $f_t(D_\varepsilon^{k-1})$ along parallel to the l_α-axis, so that its boundary $f_t(S_\varepsilon^{k-2})$ stays fixed. This extends to give an isotopy $f_{t/(D_i^{k-1} \times 0)}$, $0 \le t \le \tfrac{1}{2}$, with support contained in $(\operatorname{int} D_\varepsilon^{k-1})$ such that $f_{\frac{1}{2}}(D_i^{k-1} \times 0)$ is linked with $(0 \times D_j^{n-k})$.

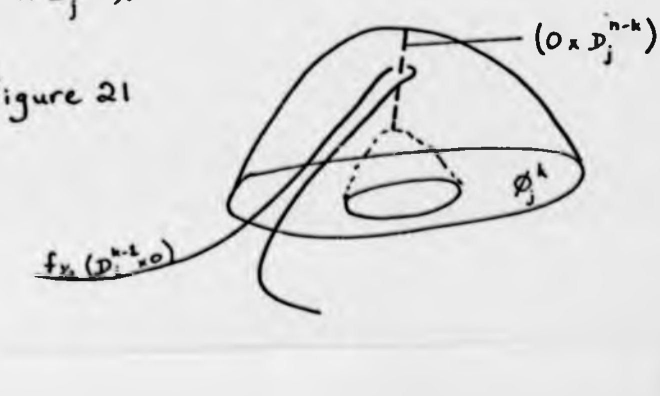


figure 21

Provided $k < n$, we can put a twist in $f_t(D_i^{k-1} \times 0)$ as we pull it across the tube $N_\varepsilon(l_\alpha)$ in order to create local linking number $\varepsilon = \pm 1$ with $(0 \times D_j^{n-k})$. Recall that $\sigma_{i,c}^{k-1}$ is the chosen orientation of $(D_i^{k-1} \times 0)$ and $\sigma_{j,c}^{k}$ that of $(D_j^{k} \times 0)$. Let $\eta_{j,p}$ be the outward normal to $\partial(D_j^k \times 0) = (S_j^{k-1} \times 0)$ at p. The induced orientation $\dot{\sigma}_{j,c}^{k-1}$ of $(S_j^{k-1} \times 0)$ is defined by $\sigma_{j,c}^{k} = [\eta_{j,p}, \dot{\sigma}_{j,c}^{k-1}]$.

If $f_{\frac{1}{2}}(D_i^k \times 0)$ is linked with $(0 \times D_j^{n-k})$ then the local linking number created will be $\varepsilon = \langle f_{\frac{1}{2}}(\sigma_{i,c}^{k-1}), \dot{\sigma}_{j,c}^{k-1} \rangle$.

(To ensure that this comparison is well defined, when $k = 2$ we require that $N_\varepsilon(l_\alpha)$ enter φ_j^k through a single component of $(S_j^{k-1} \times D_j^{n-k}) \cap \partial \varphi^0$.)

To obtain $f_1 \in T_{q_t}$ we push $f_t(D_i^{k-1} \times 0)$ back into M_{k-1} by radial isotopy defined on $\varphi_j^k \setminus (0 \times D_j^{n-k})$. By the extension of isotopy theorem we obtain an isotopy f_t, $0 \le t \le 1$, with support contained in $f^{-1}(\text{int } M_k)$, such that the images of the core discs $f(D_l^m \times 0)$ $m \le k-1$ $l \ne i$ are unchanged. Also, $f_t(M_m) \subset \text{int } M_m$ for $m \ge k$, so $f_{\#_m}$ is unchanged for $m \ge k+1$. It is easy to verify, using the path based cells, that $(f_1)_{\#_{k-1}}[\varphi_i^{k-1}]$ and $(f_1)_{\#_k}$ have the required forms. Suppose that as $f_t(D_i^{k-1} \times o)$ is pushed down out of φ_j^k it coincides with $(S_j^{k-1} \times 0)$. For each point $q \in \varphi_j^k(S_j^{k-1} \times 0) \pitchfork (0 \times S_l^{n-k})$ one new point $q' \in f_t(D_i^{k-1} \times 0) \pitchfork (0 \times D_l^{n-k+1})$ will be created. It follows that

$$(f_1)_{\#_{k-1}}[\varphi_i^{k-1}] - (f_0)_{\#_{k-1}}[\varphi_i^{k-1}] = \epsilon\, \alpha\, d_k[\varphi_j^k]$$
$$= d_k\, \xi_k[\varphi_i^{k-1}].$$

If φ_m^k is a k-handle whose attaching map hits φ_i^{k-1}, then during the isotopy f_t $0 \leq t \leq 1$ $f_t(D_m^k \times 0)$ is pulled along the tube $N_\epsilon(l_\alpha)$ behind $f_t(D_i^{k-1} \times 0)$. (Recall that $\mathcal{H}$ is fitted so φ_m^k is attached onto complete core discs of φ_i^{k-1}).

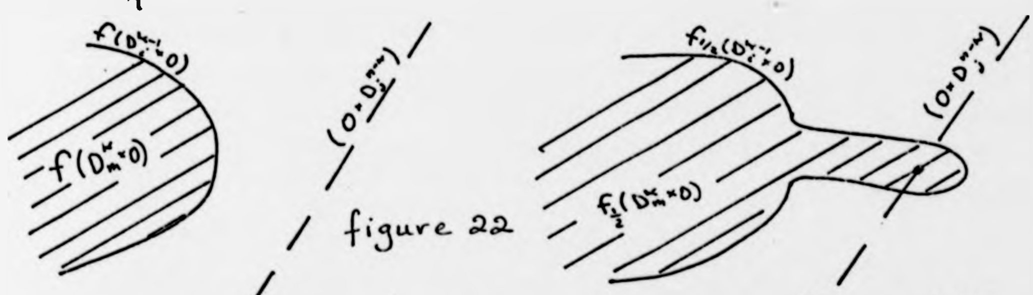


figure 22

Therefore one new point $q' \in f_1(D_m^k \times 0) \pitchfork (0 \times D_j^{n-k})$ is created for each point $q \in \varphi_m^k(S_m^{k-1} \times 0) \pitchfork (0 \times S_i^{n-k})$. Taking account of the images of the base paths $f_t(\Gamma_m^k)$ and $f_t(\Gamma_i^{k-1})$ it follows that

$$(f_1)_{\#_k}[\varphi_m^k] - (f_0)_{\#_k}[\varphi_m^k] = f_*(d_{mi}^k) \quad \epsilon\, \alpha\, [\varphi_j^k]$$
$$= \xi_k\, d_k[\varphi_m^k]$$

where $f_* : \pi_1(M, x_0) \circlearrowleft$ is unchanged and

$$d_k[\varphi_m^k] = \sum_{h=1}^{r_{k-1}} d_{mh}^k[\varphi_h^{k-1}], \quad d_{mh}^k \in Z[\pi_1] \quad (k \geq 3).$$

This completes the proof of (2.8) when $3 \leq k \leq n-1$.

<u>Case 2</u> $\xi_2 : \rho_1 \to \rho_2$. Suppose $\xi_2([\varphi_i^1]) = \epsilon\, x [\varphi_j^2]$ where $\epsilon = \pm 1$ and $x \in \rho_1$. We take hold of a small arc in $f(D_i^1 \times 0) \cap (\text{int}\ \varphi^0)$ and pull a loop first through the word in (1)-handles, $x \in \rho_1$, then along the base path Γ_j^2 and finally into φ_j^2 through $[z_0^j]$ and create linking number ϵ with $(0 \times D_j^{n-2})$.

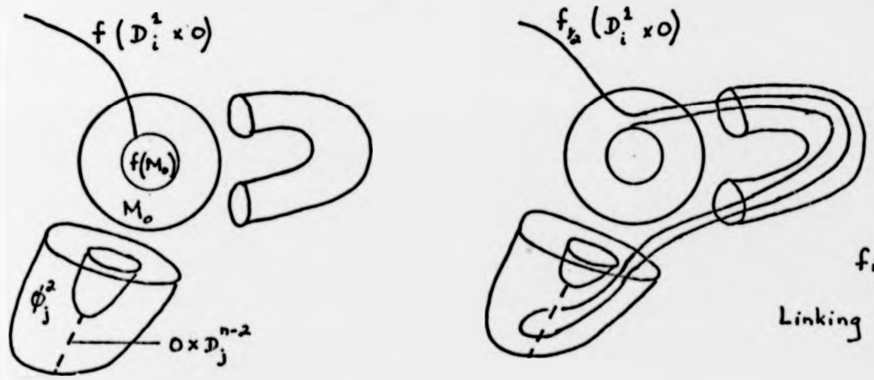


figure 23
Linking $\xi_2[\phi_i^1] = \alpha[\phi_j^2]$

After pushing $f_t(D_i^1 \times 0)$ back down into int M_1 we obtain:

$$(f_1)_{\#_1}[\varphi_i^1] = x\, d_2([\varphi_j^2])^{\epsilon}\, x^{-1}\, (f_0)_{\#_1}([\varphi_i^1]) \quad \text{as}$$

required.

We have to check that this has the required effect on $f_{\#_2}([\varphi_k^2])$ when $[\varphi_i^1]$ appears in $d_2([\varphi_k^2])$. Suppose $d_2([\varphi_k^2]) = [\varphi_l^1][\varphi_i^1]$ and that we create linking number $\xi_2([\varphi_i^1]) = \epsilon\, x\, [\varphi_j^2]$ as above. Then the isotopy will also create a new point of intersection in $f(D_k^2 \times 0) \pitchfork (0 \times D_j^{n-2})$.

This point will be associated with $(f_{\#_1}([\varphi_i^1])\,x) \in \rho_1$ as in the following pictures:

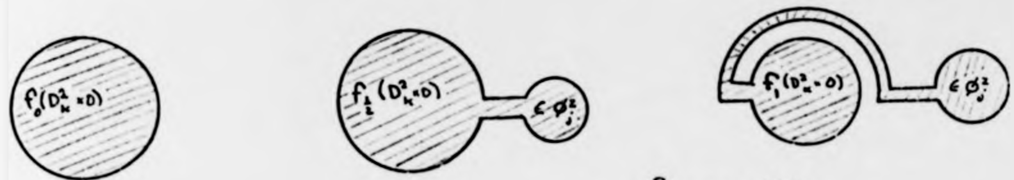


figure 24
effect of ξ_2 on $f_{\#_2}$

Therefore $(f_1)_{\#_2}([\varphi_k^2]) = (f_0)_{\#_1}([\varphi_i^1])\,\xi_2([\varphi_i^1]) + (f_0)_{\#_2}([\varphi_k^2])$

Since $\xi_2\colon \rho_1 \to \rho_2$ is defined to be a crossed homomorphism associated with $(f_0)_{\#_1}$ it follows that $(f_1)_{\#_2} - (f_0)_{\#_2} = \xi_2 d_2$

as required.

Case 3 It remains to realize $\omega \in \rho_1$. We pull the image $f_t(x_0)$ through the sequence of oriented (1)-handles given by the word $\omega = [\varphi_{i_1}^1]^{\varepsilon_1} \dots [\varphi_{i_k}^1]^{\varepsilon_k}$ $\varepsilon_j = \pm 1$.

It follows that for any (1)-handle φ_i^1

$(f_1)_{\#_1}[\varphi_i^1] = \omega^{-1}(f_0)_{\#_1}[\varphi_i^1]\,\omega$.

When we extend the isotopy to M, the images of all the base paths $f_t(\Gamma_i^k)$ are also pulled through $\omega \in \rho_1$. Therefore

$$(f_1)_{\#_2}[\varphi_i^2] = \omega^{-1}(f_0)_{\#_2}[\varphi_i^2]$$

and $(f_1)_{\#_k}([\varphi_i^k]) = (\bar{\omega})^{-1}(f_0)_{\#_k}([\varphi_i^k])$ for $k \geq 3$

as required.

This completes the proof of (2.7) when $\xi_n = 0$. $\square$

<u>Remark</u> Following Whitehead we have defined chain homotopy so that when $E \underset{\xi,\omega}{\simeq} E'$

$$\bar{\omega}\, E_k' - E_k = d_{k+1}\, \xi_{k+1} + \xi_k\, d_k .$$

Therefore when we realize such a chain homotopy (ξ, ω), either we realize $\omega \in \rho_1$ last, or else create linking numbers $(\bar{\omega}^{-1}\xi_{ij}^k)$, $k \geq 3$ and $(\omega^{-1}\xi_{ij}^2)$ $k = 2$.

2.9) When $k = n$ the previous argument breaks down. Suppose $\xi_n[\varphi_i^{n-1}] = \epsilon\beta[\varphi_1^n]$, $\epsilon = \pm 1$, $\beta \in \pi_1(M, x_0)$ and let l_β be an arc such that $[f(\Gamma_i^{n-1})\, l_\beta\, (\Gamma_1^{n-1})^{-1}] = \beta$.(†) Then l_β can meet $f(D_k^{n-1} \times 0)$ $k \neq i$ in general position.

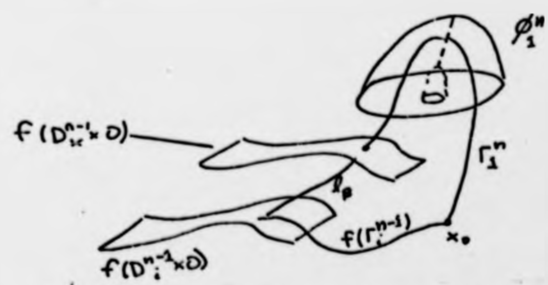


Figure 25
l_β meets $f(D_k^{n-1}\times 0)$ in general position

If we used such an arc in the procedure of (2.8) then $f_t(D_k^{n-1} \times 0)$ would also be linked with $(0 \times D_1^0)$. We can also see in another way that some limitation must be imposed on $\xi_n: \rho_{n-1} \to \rho_n$. If $g: M \to M$ is a cellular homeomorphism, then $g_{\#_n}[\varphi_1^n]$ is necessarily of the form $\pm\gamma[\varphi_1^n]$ for some $\gamma \in \pi_1(M, x_0)$. We are calling homomorphisms

(†) Recall that $\mathcal{N}$ has a single n-handle $\varphi_1^n = (D_1^n \times D_1^0)$.

$E: \rho(M, \mathcal{H})$ having this property 'monic' by anology with monic polynomials. On the other hand any $\xi_n: \rho_{n-1} \to \rho_n$ that can be realized by an isotopy of f must correspond to a word $\omega' \in \pi_1(M_1', x_0')$, the free group of the dual decomposition. Let $x_0' = (0 \times D_1^0)$. When $f_t(D_i^{n-1} \times 0)$ is pulled across $(0 \times D_1^0)$, the inverse image $f_t^{-1}(x_0')$ is being pulled through the dual (1)-handle $(\varphi_i^{1\prime})$.

We assume dimension $M \geq 4$, so it is equivalent to consider $\xi_{*n}: C_{n-1} \to C_n$. Given ξ_* on $\tilde{C}(M, \mathcal{H})$ we construct a chain homotopy $\mathcal{D}(\xi_*)$ on $\tilde{C}(M, \mathcal{H}')$. Using (2.5) we then lift $\mathcal{D}(\xi_*)$ to $\rho(M, \mathcal{H}')$. When $f_{\#} \simeq E_{\xi,\omega}$ and E is monic this procedure will show that the required word $\omega' \in \pi_1(M_1', x_0')$ corresponding to ξ_n does exist.

We first determine the chain complex $C' = \tilde{C}(M, \mathcal{H}')$. For the time being, we will keep track of base points and regard C' as a complex over $Z[\pi_1(M, x_0')]$. Clearly, C'_{n-k} is a free module of the same rank as C_k. Choose a basis of path based cells for C'_{n-k} as follows. Let the base path Γ_1^n of φ_1^n be a path from x_0 to x_0' and write γ for $(\Gamma_1^n)^{-1}$. The base path $\Gamma_i'^{n-k}$ of $\varphi_i'^{n-k}$ will be $(\gamma\Gamma_i^k)$. Recall that $ORE(\varphi_i^k) = [\sigma_{i,c}^k, \sigma_{i,t}^{n-k}]$. Let the $\sigma_{i,t}^{n-k}$ orient the dual core discs $(0 \times D_i^{n-k})$. The orientation $\dot{\sigma}_{i,t}^{n-k-1}$ of $(0 \times S_i^{n-k-1})$ is defined by $\sigma_{i,t,p}^{n-k} = [\eta_{i,p}, \dot{\sigma}_{i,t,p}^{n-k-1}]$ where $\eta_{i,p}$ is the outward normal at p.

We define a map $\delta: \pi_1(M, x_0) \to \{+1, -1\}$ as follows:

for $\alpha \in \pi_1(M, x_0)$

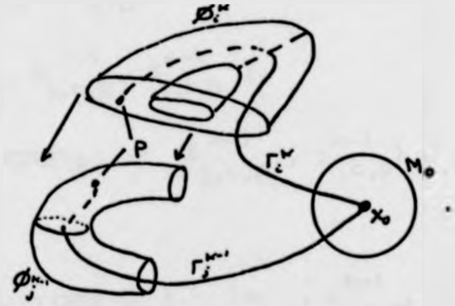
$$\delta_\alpha = +1 \quad \text{if } \alpha \text{ is orientation preserving}$$
$$= -1 \quad \text{if not.}$$

Let $(\quad)^* : Z[\pi_1(M, x_0)] \circlearrowleft$ be the map $\sum_\alpha k_\alpha \alpha \to \sum_\alpha k_\alpha \delta_\alpha (\alpha)^{-1}$. $(\quad)^*$ is an antiautomorphism of period 2 i.e. $(\alpha\beta)^* = \beta^* \alpha^*$ and $(\alpha^*)^* = \alpha$. [12 Page 398]. Notice that if $p \in \varphi_i^k(S_i^{k-1} \times 0) \pitchfork (0 \times S_j^{n-k})$ then we can compare the orientations of the two handles at p. It follows from the definitions that

$$\langle \text{ORE } \varphi_i^k, \text{ ORE } \varphi_j^{k-1} \rangle_p = \delta_{\alpha(p)} .$$

figure 26

$\delta_\alpha(p) = \langle \text{ORE } \phi_i^k, \text{ ORE } \phi_j^{k-1} \rangle_p$



Let $\Phi_\gamma : \pi_1(M, x_0) \to \pi_1(M, x_0')$ be the isomorphism $\alpha \to \gamma \alpha \gamma^{-1}$. We now calculate the boundary map of $\tilde{C}(M, \mathfrak{M}')$.

$$C_k \xrightarrow{\partial_k} C_{k-1}$$

$$C'_{n-k} \xleftarrow[\partial'_{n-k+1}]{} C'_{n-k+1}$$

2.10) <u>Lemma</u> $\quad \partial'^{\,n-k+1}_{ij} = (-1)^k \, \Phi_\gamma((\partial^k_{ji})^*)$ (+)

<u>Proof</u> We have $\partial^k_{ji} = \sum\limits_{p \in \varphi^k_j(S^{k-1}_j \times 0) \pitchfork (0 \times S^{n-k}_i)} \mathrm{ORE}(p)\ \alpha(p)$.

When M is turned upside down, the two spheres exchange roles but the intersection is unchanged. Let $\mathrm{ORE}'(p) = \langle \dot\delta^{n-k}_{i,t}, \sigma^{n-k}_{j,t} \rangle$ and $\alpha'(p) = [(\Gamma'^{\,n-k+1}_i |_p)(\Gamma'^{\,n-k}_j |_p)^{-1}] \in \pi_1(M, x_0')$.

We have

$$\mathrm{ORE}\,(\varphi^{k-1}_i)_p = [\sigma^{k-1}_{i,c,p}, \eta_{i,p}, \dot\sigma^{n-k}_{i,t,p}]$$

$$= \delta_{\alpha(p)}\, \mathrm{ORE}\,(\varphi^k_j)_p$$

$$= \delta_{\alpha(p)}\, [\sigma^k_{j,c,p}, \sigma^{n-k}_{j,t,p}]$$

$$= \delta_{\alpha(p)}\, [\eta_{j,p}, \dot\sigma^{k-1}_{j,c,p}, \sigma^{n-k}_{j,t,p}].$$

But $\eta_{i,p} = (-1)\,\eta_{j,p}$ and $\mathrm{ORE}(p) = \langle \dot\sigma^{k-1}_{j,c,p}, \sigma^{k-1}_{i,c,p} \rangle$.

Substituting we get:

$$[\sigma^{k-1}_{i,c,p}, \eta_{i,p}, \dot\sigma^{n-k}_{i,t,p}] = (-1)\,\mathrm{ORE}(p)\ \delta_{\alpha(p)}[\eta_{i,p}, \sigma^{k-1}_{i,c,p}, \sigma^{n-k}_{j,t,p}]$$

$$= (-1)^k\,\mathrm{ORE}(p)\ \delta_{\alpha(p)}[\sigma^{k-1}_{i,c,p}, \eta_{i,p}, \sigma^{n-k}_{j,t,p}].$$

Therefore $\langle \dot\sigma^{n-k}_{i,t,p}, \sigma^{n-k}_{j,t,p} \rangle = (-1)^k\,\mathrm{ORE}(p)\ \delta_{\alpha(p)}$.

(+) Without reference to base-points this Lemma is proved in [12] page 395.

Similarly $\alpha'(p) = [\Gamma_i'^{n-k+1}|_p)(\Gamma_j'^{n-k}|_p)^{-1}]$

$$= [(\gamma\Gamma_i^{k-1}|_p)(\gamma\Gamma_j^{k}|_p)^{-1}]$$

$$= [\gamma(\Gamma_i^{k-1}|_p)(\Gamma_j^{k}|_p)^{-1}\ \gamma^{-1}]$$

$$= \gamma(\alpha(p))^{-1}\ \gamma^{-1}.$$

Therefore

$$\partial_{ij}'^{n-k+1} = \sum_{p\in\varphi_j^k(S_j^{k-1}\times 0)\pitchfork(0\times S_i^{n-k})} (-1)^k \,\mathrm{ORE}(p)\ \delta_{\alpha(p)}\ \Phi_\gamma(\alpha(p)^{-1})$$

$$= (-1)^k\ \Phi_\gamma((\partial_{ji}^k)^*)\ . \quad \square$$

If $f \in T_\varphi$ then $(f^{-1}) \in T_{\varphi'}$. Given the matrix $(\hat{A}_k) = (\hat{a}_{ij}^k)$ representing $f_{*k} : \tilde{C}_k \circlearrowleft$ we calculate $(\hat{A}'_{n-k}) = (\hat{a}_{ij}'^{n-k})$ representing $(f^{-1})_{*n-k} : C'_{n-k} \circlearrowleft$.

Let $\hat{a}_{11}^n = \epsilon_n\ \alpha_n$, so $\alpha_n = [f(\Gamma_1^n)(\Gamma_1^n)^{-1}]$. Therefore

$$\langle f(\mathrm{ORE}\ \varphi^0), \mathrm{ORE}\ \varphi^0\rangle_{x_0} = \delta_{\alpha_n}\langle f(\mathrm{ORE}\ \varphi_1^n), \mathrm{ORE}\ \varphi_1^n\rangle_{x_0'}$$

$$= \delta_{\alpha_n}\langle f(\sigma_{1,c}^n), \sigma_{1,c}^n\rangle_{x_0'}$$

$$= \delta_{\alpha_n}\ \epsilon_n\ .$$

Similarly if $p \in f(D_i^k \times 0)\pitchfork(0\times D_j^{n-k})$ then

$$\langle f(\mathrm{ORE}\ \varphi_i^k), \mathrm{ORE}\ \varphi_j^k\rangle_p = \delta_{\alpha(p)}\langle f(\mathrm{ORE}\ \varphi^0), \mathrm{ORE}\ \varphi^0\rangle_{x_0}$$

$$= \delta_{\alpha(p)}\ \epsilon_n\ \delta_{\alpha_n}\ .$$

2.11) <u>Lemma</u> $(\hat{a}'^{\,n-k}_{ij}) = \epsilon_n\, \delta_{\alpha_n}\, \Phi_\gamma(f_*^{-1}(\alpha_n(\hat{a}^k_{ji})^*))$

where $f_*: \pi_1(M, x_0) \circlearrowleft$.

<u>Proof</u> $$\hat{a}'^{\,n-k}_{ij} = \sum_{q\in(D^k_j \times 0)\pitchfork f^{-1}(0\times D^{n-k}_i)} ORE'(q)\,\alpha'(q)$$

$$= \sum_{p\in f(D^k_j \times 0)\pitchfork(0\times D^{n-k}_i)} ORE'(f^{-1}(p))\,\alpha'(f^{-1}(p))$$

Now $$f(ORE(\varphi^k_j))_p = [\, f(\sigma^k_{j,c}),\, f(\sigma^{n-k}_{j,t})\,]_p$$

$$= \epsilon_n\, \delta_{\alpha_n}\, \delta_{\alpha(p)}\, (ORE\ \varphi^k_i)_p$$

$$= \epsilon_n\, \delta_{\alpha_n}\, \delta_{\alpha(p)}\, [\sigma^k_{i,c}\,,\ \sigma^{n-k}_{i,t}\,]_p$$

$$= \epsilon_n\, \delta_{\alpha_n}\, \delta_{\alpha(p)}\, ORE(p)\,[\, f(\sigma^k_{j,c}),\ \sigma^{n-k}_{i,t}\,]_p\ .$$

Since comparison is invariant under f^{-1} we get

$$ORE'(f^{-1}(p)) = \langle f^{-1}(\sigma^{n-k}_{i,t}),\ \sigma^{n-k}_{j,t}\,\rangle_{f^{-1}(p)}$$

$$= \langle \sigma^{n-k}_{i,t},\ f(\sigma^{n-k}_{j,t})\,\rangle_p$$

$$= \epsilon_n\, \delta_{\alpha_n}\, \delta_{\alpha(p)}\ ORE(p)\ .$$

$$\alpha'(f^{-1}(p)) = [\, f^{-1}(\gamma\,\Gamma^k_i)\,|_{f^{-1}(p)}\,(\gamma\,\Gamma^k_j\,|_{f^{-1}(p)})^{-1}\,]$$

$$= [\, f^{-1}(\gamma)\, f^{-1}(\alpha(p)^{-1})\,\gamma^{-1}\,].$$

We obtain a more useful expression as follows. Notice that $\Phi_\gamma(f_*^{-1}(\alpha_n)) = [\, f^{-1}(\gamma)\,\gamma^{-1}\,]$.

Therefore $\alpha'(f^{-1}(p)) = [f^{-1}(\gamma)\gamma^{-1}\ \gamma f^{-1}(\alpha(p)^{-1})\gamma^{-1}]$

$$= \Phi_\gamma(f_*^{-1}(\alpha_n)\ f_*^{-1}(\alpha(p)^{-1}) .$$

Since $\delta_{\alpha(p)}(\alpha(p))^{-1} = \alpha(p)^*$ we obtain

$$\tilde{a}'^{\,n-k}_{i,j} = \epsilon_n \delta_{\alpha(n)} \Phi_\gamma (f_*^{-1}(\alpha_n(\tilde{a}^k_{ji})^*)) . \ \square$$

We will write the induced map of (f^{-1}) on $\pi_1(M, x_0')$ as $(f^{-1})_*'$ to distinguish it from $f_*^{-1}: \pi_1(M, x_0)$. If $\bar{y} \in \pi_1(M, x_0')$ then $(f^{-1})_*'(\bar{y}) = \Phi_\gamma[f_*^{-1}(\alpha_n \Phi_{\gamma^{-1}}(\bar{y})\alpha_n^{-1})]$.

2.12) <u>Definition</u> Let $F: \check{C}(M, \mathcal{H})\circlearrowleft$ be a chain map associated with $\theta: \pi_1(M, x_0)\circlearrowleft$, represented by matrices (F^k_{ij}). Suppose F is monic and θ is an isomorphism which preserves the orientation preserving subgroup of $\pi_1(M, x_0)$. Let $F^n_{11} = \epsilon_n \alpha_n$. We define a chain map $\mathcal{D}(F)$ on $\check{C}(M, \mathcal{H}')$ by

$$\mathcal{D}(F)^{n-k}_{i,j} = \epsilon_n \delta_{\alpha_n} \Phi_\gamma [\theta^{-1}(\alpha_n(F^k_{j,i})^*]$$

and $\mathcal{D}(F)$ is associated with the isomorphism

$(\theta^{-1})': \pi_1(M, x_0')$ given by $\bar{y} \to \Phi_\gamma[\theta^{-1}(\alpha_n \Phi_{\gamma^{-1}}(\bar{y})\alpha_n^{-1})]$

for $\bar{y} \in \pi_1(M, x_0')$. (†)

Notice that if $f \in T_{\mathcal{H}}$ and $f_*: \check{C}(M, \mathcal{H})\circlearrowleft$ then $(f^{-1})_* = \mathcal{D}(f_*)$. Also if $F, G: \check{C}(M, \mathcal{H})\circlearrowleft$ are as above and $\mathcal{D}(F) = \mathcal{D}(G)$ then $F = G$.

Let $G: \check{C}(M, \mathcal{H})\circlearrowleft$ be a monic chain map associated with $\psi: \pi_1(M, x_0)\circlearrowleft$ and suppose that $F \underset{(\eta, \bar{\omega})}{\simeq} G$. It follows

(†) **Disregarding base points $\mathcal{D}$ is essentially the functor $\mathrm{Hom}_{Z[\pi_1]}(\ \ , Z[\pi_1])$. See (3.14) below.**

that for $\bar{x} \in \pi_1(M, x_0)$ $\psi(\bar{x}) = \bar{\omega}^{-1}\theta(\bar{x})\bar{\omega}$, so ψ is also an isomorphism which preserves the orientation preserving subgroup. Let $G_{11}^n = \mu_n \beta_n$, $\mu_n = \pm 1$, $\beta_n \in \pi_1(M, x_0)$. Define a chain homotopy $\mathcal{D}(\eta, \bar{\omega})$ on $\tilde{C}(M, \mathcal{H}')$ associated with $(\theta^{-1})'$ by

$$\mathcal{D}(\bar{\omega}) = \Phi_\gamma [\theta^{-1}(\alpha_n \beta_n^{-1} \bar{\omega}^{-1})]$$

$$\mathcal{D}(\eta)_{i,j}^{n-k+1} = (-1)^k \epsilon_n \delta_{\alpha_n} \Phi_\gamma [\theta^{-1}(\alpha_n(\eta_{j,i}^k)^*)].$$

Remark Let $f \in T_{\mathcal{H}}$ and suppose $f_* \underset{\eta,\bar{\omega}}{\simeq} G$ as above. If f_t $0 \le t \le 1$ is an isotopy which realizes $(\eta, \bar{\omega})$ on $\tilde{C}(M, \mathcal{H})$ then one can prove as in (2.10) and (2.11) that f_t^{-1} $0 \le t \le 1$ realizes $\mathcal{D}(\eta, \bar{\omega})$ on $\tilde{C}(M, \mathcal{H}')$.

2.13) Lemma Suppose $F, G: \tilde{C}(M, \mathcal{H})\circlearrowleft$ are as in (2.12) If $F \underset{\eta,\omega}{\simeq} G$, then $\mathcal{D}(F) \underset{\mathcal{D}(\eta,\bar{\omega})}{\simeq} \mathcal{D}(G)$.

Proof We will need to know that $\mu_n \delta_{\beta_n} \delta_{\bar{\omega}} = \epsilon_n \delta_{\alpha_n}$.

Define a map $\bar{\delta}: Z[\pi_1] \to Z$ by $\bar{\delta}(\sum_\alpha k_\alpha \alpha) = \sum_\alpha k_\alpha \delta_\alpha$. $\bar{\delta}$ is a homomorphism of the additive group structures. Since $F \underset{\eta,\omega}{\simeq} G$, we have

$$(\bar{\omega}\mu_n \beta_n - \epsilon_n \alpha_n)\varphi_1^n = \eta_n \partial_n(\varphi_1^n)$$

$$= \sum_{h=1}^{r_{n-1}} \theta(\partial_{1h}^n)\eta_{h1}^n$$

Since $\mathcal{H}$ has a single (n)-handle, $\varphi_1^n(S_1^{n-1} \times 0) \pitchfork (0 \times S_h^0)$ consists of two points, so we can write

$$\partial^n_{1,h} = (\epsilon_{h_1} x_{h_1} + \epsilon_{h_2} x_{h_2}) \qquad \epsilon_{h_j} = \pm 1$$

and $x_{h_j} \in \pi_1(M, x_0)$.

Therefore

$$\begin{aligned}\bar{\delta}(\bar{\omega}\,\mu_n \beta_n - \epsilon_n \alpha_n) &= \mu_n \delta_{\bar{\omega}} \delta_{\beta_n} - \epsilon_n \delta_{\alpha_n} \\ &= \bar{\delta}\Big(\sum_{h=1}^{r_{n-1}} \theta(\partial^n_{1h})\,\eta^n_{h,1}\Big) \\ &= \sum_{h=1}^{r_{n-1}} (\epsilon_{h_1} \delta_{\theta(x_{h_1})} + \epsilon_{h_2} \delta_{\theta(x_{h_2})})\,\bar{\delta}(\eta^n_{h,}\end{aligned}$$

and $\delta_{\theta(x_{h_j})} = \delta_{x_{h_j}}$.

We claim $(\epsilon_{h_1} \delta_{x_{h_1}} + \epsilon_{h_2} \delta_{x_{h_2}}) = 0$ for all $h = 1, \ldots, r_{n-1}$.

Suppose the dual (1)-handle $(\varphi^1_h)'$ is orientation preserving. Then $\delta_{x_{h_1}} = \delta_{x_{h_2}}$ and $\epsilon_{h_1} = -\epsilon_{h_2}$. If $(\varphi^1_h)'$ is not orientation preserving then $\delta_{x_{h_1}} = -\delta_{x_{h_2}}$ but $\epsilon_{h_1} = -\epsilon_{h_2}$.

Therefore $\mu_n \delta_{\bar{\omega}} \delta_{\beta_n} - \epsilon_n \delta_{\alpha_n} = 0$.

We now prove the lemma by calculation. We have $\bar{\omega}\, G_k - F_k = \partial_{k+1} \eta_{k+1} + \eta_k \partial_k$. Therefore

$$(1) \quad (\bar{\omega}\, G^k_{i,j} - F^k_{i,j}) = \sum_{l=1}^{r_{k+1}} \eta^{k+1}_{il}\, \partial^{k+1}_{lj} + \sum_{h=1}^{r_{k-1}} \theta(\partial^k_{ih})\, \eta^k_{hj}.$$

We have to show

$$\mathcal{D}(\bar{\omega})\,\mathcal{D}(G)_{n-k} - \mathcal{D}(F)_{n-k} = \partial'_{n-k+1}\,\mathcal{D}(\eta)_{n-k+1} + \mathcal{D}(\eta)_{n-k}\,\partial'_{n-k}.$$

We show the matrix entries satisfy:

$$(2)\qquad \mathcal{D}(\bar{\omega})\,\mathcal{D}(G)^{n-k}_{ji} - \mathcal{D}(F)^{n-k}_{ji} = \sum_{h=1}^{r_{k-1}} \mathcal{D}(\eta)^{n-k+1}_{jh}\,\partial'^{\,n-k+1}_{hi}$$

$$+ \sum_{l=1}^{r_{k+1}} (\theta^{-1})'(\partial'^{\,n-k}_{jl})\,\mathcal{D}(\eta)^{n-k}_{li} \quad .$$

From the definitions (2.12), the left hand side of (2) is:

$$\text{L.H.S.} = \Phi_\gamma(\theta^{-1}(\alpha_n\,\beta_n^{-1}\,\bar{\omega}^{-1}))(\mu_n\,\delta_{\beta_n}\Phi_\gamma(\psi^{-1}(\beta_n(G^k_{ij})^*))$$

$$-\ \epsilon_n\,\delta_{\alpha_n}\Phi_\gamma(\theta^{-1}(\alpha_n(F^k_{ij})^*)) \ .$$

For $\bar{x} \in \pi_1(M, x_0)$ $\psi^{-1}(\bar{x}) = \theta^{-1}(\bar{\omega})\,\theta^{-1}(\bar{x})\,\theta^{-1}(\bar{\omega}^{-1})$.

Therefore

$$\text{L.H.S.} = \mu_n\,\delta_{\beta_n}\,\Phi_\gamma[\,\theta^{-1}(\alpha_n\,\beta_n^{-1}\bar{\omega}^{-1})\,\theta^{-1}(\bar{\omega})\,\theta^{-1}(\beta_n)\theta^{-1}(G^k_{ij})^*\,\theta^{-1}(\bar{\omega}^{-1})\,]$$

$$-\ \epsilon_n\,\delta_{\alpha_n}\,\Phi_\gamma\,[\,\theta^{-1}(\alpha_n(F^k_{ij})^*)\,]$$

$$= \mu_n\,\delta_{\beta_n}\,\delta_{\bar{\omega}}\,\Phi_\gamma(\theta^{-1}(\alpha_n(\bar{\omega}G^k_{ij})^*))$$

$$-\ \epsilon_n\,\delta_{\alpha_n}\,\Phi_\gamma(\theta^{-1}(\alpha_n(F^k_{ij})^*)) \ .$$

Therefore

$$\text{L.H.S.} = \epsilon_n\,\delta_{\alpha_n}\,\Phi_\gamma(\theta^{-1}(\alpha_n(\bar{\omega}\,G^k_{ij} - F^k_{ij})^*)) \ . \qquad (3)$$

Similarly we calculate each summand in the right hand side of (2).

$$\sum_{h=1}^{r_{k-1}} \mathcal{D}(\eta)^{n-k+1}_{jh}\,\partial'^{\,n-k+1}_{hi} = \sum_{h=1}^{r_{k-1}} (-1)^k\epsilon_n\delta_{\alpha_n}\Phi_\gamma[\theta^{-1}(\alpha_n(\eta^k_{hj})^*)](-1)^k\Phi_\gamma(\partial^k_{ih})^*$$

$$= \epsilon_n\delta_{\alpha_n}\sum_{h=1}^{r_{k-1}}\Phi_\gamma[\theta^{-1}(\alpha_n)\theta^{-1}(\eta^k_{hj})^*(\partial^k_{ih})^*\,]$$

but $\delta_{\theta^{-1}(\bar{x})} = \delta_{\bar{x}}$ and $\alpha^* \beta^* = (\beta\alpha)^*$, so

$$= \epsilon_n \delta_{\alpha_n} \sum_{h=1}^{r_{k-1}} \Phi_\gamma(\theta^{-1}(\alpha_n(\theta(\partial^k_{in})\,\eta^k_{nj})^*))$$

$$= \epsilon_n \delta_{\alpha_n} \Phi_\gamma[\theta^{-1}(\alpha_n(\sum_{h=1}^{r_{k-1}} \theta(\partial^k_{in})\,\eta^k_{nj})^*)]$$

and $\sum_{l=1}^{r_{k+1}} (\theta^{-1})'\,(\partial'^{\,n-k}_{jl})\,\mathcal{D}(\eta)^{n-k}_{li}$

$$= \sum_{l=1}^{r_{k+1}} \Phi_\gamma(\theta(\alpha_n))\,\Phi_\gamma(\theta^{-1}(\Phi_\gamma^{-1}((-1)^{k+1}\Phi_\gamma(\partial^{k+1}_{lj})^*)))$$

$$\cdot\, \Phi_\gamma(\theta^{-1}(\alpha_n))^{-1}\;\epsilon_n \delta_{\alpha_n} (-1)^{k+1} \Phi_\gamma(\theta^{-1}(\alpha_n))$$

$$\cdot\, \Phi_\gamma(\theta^{-1}(\eta^{k+1}_{il})^*))$$

$$= \epsilon_n \delta_{\alpha_n} \sum_{l=1}^{r_{k+1}} \Phi_\gamma(\theta^{-1}(\alpha_n(\partial^{k+1}_{lj})^*(\eta^{k+1}_{il})^*))$$

$$= \epsilon_n \delta_{\alpha_n} \Phi_\gamma\,[\theta^{-1}(\alpha_n(\sum_{l=1}^{r_{k+1}} \eta^{k+1}_{il}\,\partial^{k+1}_{lj})^*)]$$

Thus

$$\text{L.H.S.} = \epsilon_n \delta_{\alpha_n} \Phi_\gamma[\theta^{-1}(\alpha_n(\bar{\omega} G^k_{ij} - F^k_{ij})^*)]$$

$$\text{R.H.S.} = \epsilon_n \delta_{\alpha_n} \Phi_\gamma[\theta^{-1}(\alpha_n(\sum_{l=1}^{r_{k+1}} \eta^{k+1}_{il}\partial^{k+1}_{lj} + \sum_{n=1}^{r_{k-1}} \theta(\partial^k_{in})\,\eta^k_{nj})^*)]$$

and equality in (2) now follows from line (1). It follows similarly from definitions that

$$(\psi^{-1})'(\bar{x}) = \mathcal{D}(\bar{\omega})^{-1}\,(\theta^{-1})'(\bar{x})\,\mathcal{D}(\bar{\omega}).$$

Together with line (2) this shows that $\mathcal{D}(F) =_{\mathcal{D}(\eta,\bar{\omega})} \mathcal{D}(G)$

as required. □

2.14) We now complete the proof of (2.7). Recall that $\mathcal{H}$ has one (0)-handle and one (n)-handle and dimension $M \geq 4$. We suppose $f \in T_{\mathcal{H}}$, and $E: \rho(M,\mathcal{H}) \circlearrowleft$ a monic homomorphism such that $f_{\#} \underset{(\xi,\omega)}{\simeq} E$. By (2.8) there is an isotopy f_t $0 \leq t \leq 1$ which realizes $(\{\xi_k\}_{k\leq n-1}; \omega)$. It follows that $(f_1)_{\#_k} = E_k$ for $k \leq n-2$ and $E \simeq (f_1)_{\#}$ by the chain homotopy consisting of the single non-zero map $\eta_n = (\bar{\omega}^{-1}\xi_n)$. For $\bar{x} \in \pi_1(M, x_0)$ $(f_1)_*(\bar{x}) = \bar{\omega}^{-1}(f_0)_*(\bar{x})\,\bar{\omega}$. Therefore when we apply the definition (2.12) we get $\mathcal{D}(\eta, 1) = \mathcal{D}(\xi_*)_1$, $\mathcal{D}(\bar{\omega})$. By (2.13) $(f_1^{-1})_* \simeq \mathcal{D}(E_*)$ by $(\mathcal{D}(\xi_*)_1, \mathcal{D}(\bar{\omega}).)$ We verify directly that this chain homotopy has a lift to $\rho(M, \mathcal{H}')$ of the required form.

By the proof of (2.13) we have

$$\partial_1' \mathcal{D}(\xi_*)_1 = \mathcal{D}(\bar{\omega})\,\mathcal{D}(E_*)_0 - (f_1^{-1})_{*_0} .$$

Hence

$$\partial_1' \mathcal{D}(\xi_*)_1(\varphi^0)' = \mathcal{D}(\bar{\omega})\mathcal{D}(E_*)_0(\varphi^0)' - (f_1^{-1})_0(\varphi^0)'$$

$$= (\mathcal{D}(\bar{\omega}) - 1)(\varphi^0)' .$$

Whitehead proves the following fact (Therorem 8.C [25]): If $c \in \breve{C}_1$ and $\partial_1(c) = (\bar{x} - 1)\varphi^0$ for $\bar{x} \in \bar{\rho}_1$ then for some $y \in \rho_1$, $\bar{x} = \bar{y}$ and $c = h_1(y)$. Therefore in our case there exists $\omega' \in \rho_1' = \pi_1(M_1', x_0')$ such that $\bar{\omega}' = \mathcal{D}(\bar{\omega})$ and $h_1'(\omega') = \mathcal{D}(\xi_*)_1(\varphi^0)'$. Thus $(\mathcal{D}(\bar{\omega}), \mathcal{D}(\xi_*)_1)$ is the induced chain homotopy on $\breve{C}(M,\mathcal{H}')$ of $\omega' \in \rho_1'$, regarded as a chain homotopy defined on $\rho(M,\mathcal{H}')$.

By (2.8) we can realize the chain homotopy $\omega' \in \rho_1'$ by an isotopy f_t^{-1}, $1 \le t \le 2$, such that $f_t^{-1}(M_i') \subset \operatorname{int} M_i'$ for $i \ge 1$. Therefore $(f_2)_*^{-1} = \mathscr{D}(E_*)$ and $(f_2)_* = E_*$. Also, $f_t(M_{n-i-1}) \subset \operatorname{int} M_{n-i-1}$ and $f_t(x_0) = x_0$ for $1 \le t \le 2$ and $i \ge 1$. Since $n \ge 4$, this imples $(f_2)_{\#_j} = (f_1)_{\#_j} = E_j$ for $j = 1,2$. Hence $(f_2)_\# = E$. $\square$

Using the same methods we can determine simultaneously the endomorphisms we can realize by isotopy on $\pi_1(M_1', x_0')$ and $\pi_2(M_2', M_1', x_0')$.

2.15) Corollary Suppose dim $M \ge 5$, $\mathscr{U}$ a fitted handle decomposition with one (0)-handle and one (n)-handle, $f \in T_{\mathscr{U}}$. Suppose $E: \rho(M,\mathscr{U})\circlearrowleft$ is monic and $E': \rho(M,\mathscr{U}')\circlearrowleft$ is a homomorphism such that $\mathscr{D}(E_*) = (E')_*$. If $f_* \simeq E_*$ on $\mathscr{C}(M,\mathscr{U})$, then f is isotopic to $g \in T_{\mathscr{U}}$ such that $g_\# = E$ and $(g^{-1})_\# = E'$.

Proof By (2.5) $f_\# \simeq E$ on $\rho(M,\mathscr{U})$ and by (2.7) f is isotopic to $f_1 \in T_{\mathscr{U}}$ such that $(f_1)_\# = E$. Therefore $(f_1^{-1})_* = \mathscr{D}(E_*) = (E')_*$. By (2.5) again, $(f_1^{-1})_\# \simeq E'$ on $\rho(M,\mathscr{U}')$ by a chain homotopy (ξ',ω') such that $\xi_k' = 0$ for $k > 2$. By (2.7) we can realize (ξ',ω') by an isotopy f_t^{-1} $1 \le t \le 2$ such that $f_t^{-1}(M_i') \subset \operatorname{int} M_i'$ for $1 \le t \le 1$ and $i \ge 2$. Therefore $f_t(M_i) \subset \operatorname{int} M_i$ for $i \le n-3$. Let $f_2 = g$. Then $(g^{-1})_\# = E'$ so $(g)_* = E_*$. But since dim $M \ge 5$, $2 \le n-3$ so $(f_t)_{\#_1}$ and $(f_t)_{\#_2}$ are unchanged for $1 \le t \le 2$. Hence $g_* = E$ as well. □

Combining (2.15) with the cancelling lemma (1.1) we obtain a reduction to algebra of Problem II, for a fixed handle decomposition. Let $\mathcal{H}$ be a fitted handle decomposition with one (0)-handle and one (n)-handle, and suppose $f \in T_{\mathcal{H}}$. By an endomorphism of f we mean a pair of monic homomorphisms $E: \rho(M, \mathcal{H}) \circlearrowleft$ and $E': \rho(M, \mathcal{H}') \circlearrowleft$ such that $\mathcal{D}(E_*) = (E')_*$, and f_* is chain homotopic to E_* on $\tilde{C}(M, \mathcal{H})$. For $k \ge 3$ we assume the E_k are given as matrices over $Z[\pi_1]$. When $k = 1$ a non-negative integral matrix $|E_1| = (|E_1|_{i,j})$ is defined by counting occurances of $[\varphi^1_j]$ in the reduced word for $E_1([\varphi^1_i])$. When $k = 2$ we assume $E_2([\varphi^2_i])$ is given as a specific expression in $\rho_2(M, \mathcal{H})$ so that $|E_2|$ is also defined.

2.16) <u>Theorem</u> Suppose $\mathcal{H}$ is as above, $f \in T_{\mathcal{H}}$, dimension $M \ge 5$, and (E, E') is an endomorphism of f. Then f is isotopic to a diffeomorphism g fitted with respect to $\mathcal{H}$ such that

$$h(g) = \max \{ \log s(|E|), \log s(|E'|) \} . \quad \square$$

CHAPTER THREE

CHANGING HANDLE DECOMPOSITIONS

So far we have considered diffeomorphisms fitted with respect to some fixed handle decomposition. In order to generalize the reduction to algebra of Problem II (2.15) we need to characterize all the algebraic chain complexes which arise from handle decompositions of a given manifold. For simply connected manifolds of high dimension, this reduces to having the correct homology. The following theorem is essentially proved by Smale in [22, Theorem 6.1] but was apparently first stated in print in this form by Shub [19].

Theorem (Smale). Suppose M^n is a connected closed manifold, $\pi_1(M) = e$, $n = \dim M \geq 6$. Let $\mathcal{C} = (C_i, \partial_i)$ be a chain complex of finitely generated free abelian groups

$$0 \to C_n \to C_{n-1} \to \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \to C_0 \to 0$$

such that $C_1 = C_{n-1} = 0$. If $H_k(\mathcal{C}; Z) \cong H_k(M; Z)$ for all k then $\mathcal{C}$ is the chain complex of a handle decomposition of M. $\square$

In the non-simply connected case we consider complexes over $Z[\pi_1]$. It is too much to expect that a condition just on homology should suffice. Our approach will be through simple homotopy theory. We start with the chain complex of a given handle decomposition, and modify it by operations on the handles, as in the proof of the s-cobordism theorem. This leads to a characterization based on simple chain equivalence. Given an appropriate chain complex $\mathcal{C}$ and a simple chain equivalence $G: \tilde{\mathcal{C}}(M,\mathcal{H}) \to \mathcal{C}$ we can break G up into a sequence of elementary algebraic moves. We then mirror these algebraic moves by handle moves in the manifold.

In the middle dimensions we can realize elementary moves defined on the chain level. In the extreme dimensions 1, 2 and (n-1), (n-2) familiar problems arise controlling extraneous intersections in the Whitney Lemma. Here again we can avoid these difficulities provided the elementary algebraic moves are defined instead on the homotopy systems $\rho(M,\mathcal{H})$ and $\rho(M,\mathcal{H}')$. (This uses our results in Chapters One and Two).

Since we will use homotopy systems we are forced to pick base points x_0 and x_0'. The theory is easier to formulate if we fix these for good and regard our manifolds as doubly pointed spaces, (M, x_0, x_0'). By a <u>handle decomposition of (M, x_0, x_0')</u>. we will mean a handle decomposition of M with one (0) - handle containing x_0 in its interior, and one (n)-handle containing x_0' in its interior.

We now recall a little simple homotopy theory. All of this material is from Whitehead's paper "Simple Homotopy Types" [26], and Wall's paper "Formal Deformations" [24].

Recall that the Whitehead group $Wh(\pi_1)$ is an abelian group associated with $\pi_1 = \pi_1(M, x_0)$. If L is a square invertible matrix over $Z[\pi_1]$ then a torsion element $\tau(L) \in Wh(\pi_1)$ is defined. $\tau(L) = 0$ if and only if L can be reduced to an identity matrix by a finite sequence of the following operations:

(i) multiply a row by $(\pm \gamma)$, $\gamma \in \pi_1$

(ii) add one row to another row

(iii) form the direct sum with an identity matrix

By a <u>free</u> <u>$Z[\pi_1]$</u> <u>complex</u> we will mean a chain complex $\mathcal{C} = (C_i, \partial_i)$ of finite length, such that $C_i = 0$ for $i < 0$, and each C_i is a free $Z[\pi_1]$ - module of finite rank. We will call such a complex <u>based</u> if in addition each C_i has a chosen basis which is determined up to order, sign and multiplication of basis elements by elements of π_1. That is, each C_i has associated a class of <u>preferred</u> <u>bases</u>. Until the end of this chapter (3.18) we will only need to consider chain maps associated with the identity map on π_1; that is, all chain maps will be $Z[\pi_1]$ - module homomorphisms. If $\mathcal{C}$ and $\mathcal{C}'$ are based complexes, a <u>simple</u> <u>isomorphism</u> $F: \mathcal{C} \to \mathcal{C}'$ is a chain map such that for all i

$F_i : C_i \to C_i'$ is an isomorphism, and if $<F_i>$ is the matrix representing F_i with respect to some preferred bases of C_i and C_i' then $\tau(<F_i>) = 0$.

A based acyclic complex $\mathcal{B}$ is <u>elementary</u> if $B_i = 0$ except in two adjacent dimensions $r, r-1$ and $\partial_r : B_r \to B_{r-1}$ satisfies $\tau(<\partial_i>) = 0$. A complex is <u>collapsible</u> if it is a direct sum of finitely many elementary complexes.

Let $\mathcal{C}$ and $\mathcal{C}'$ be based complexes. We shall say $\mathcal{C}$ and $\mathcal{C}'$ are in the same simple equivalence class if and only if there are collapsible complexes $\mathcal{B}, \mathcal{B}'$ and a simple isomorphism $F: \mathcal{C} \oplus \mathcal{B} \to \mathcal{C}' \oplus \mathcal{B}'$.

Let $i: \mathcal{C} \to \mathcal{C} \oplus \mathcal{B}$ be the inclusion; a <u>retraction</u> $k: \mathcal{C} \oplus \mathcal{B} \to \mathcal{C}$ is a chain map such that $ki = 1_{\mathcal{C}}$. A chain map $G: \mathcal{C} \to \mathcal{C}'$ is a <u>simple chain equivalence</u> if and only if there are collapsible complexes $\mathcal{B}, \mathcal{B}'$, a simple isomorphism $F: \mathcal{C} \oplus \mathcal{B} \to \mathcal{C}' \oplus \mathcal{B}'$ and a retraction $k: \mathcal{C}' \oplus \mathcal{B}' \to \mathcal{C}'$ such that G is chain homotopic to $k \circ F \circ i$.

For any chain equivalence $G : \mathcal{C} \to \mathcal{C}'$ a torsion element $\tau(G) \in Wh(\pi_1)$ is defined using the algebraic mapping cone construction. Whitehead proved that $G : \mathcal{C} \to \mathcal{C}'$ is a simple chain equivalence if and only if $\tau(G) = 0$ [26, Theorem 9].

3.1) Proposition Let M be a closed connected manifold, x_0, x_0' base points, and $\mathcal{H}$ and $\mathcal{H}^1$ two handle decompositions of (M, x_0, x_0'). Then there exists a simple chain equivalence $\check{\mathcal{C}}(M, \mathcal{H}) \to \check{\mathcal{C}}(M, \mathcal{H}^1)$.

Proof. Let $\mathcal{H} = \{M_i, 0 \leq i \leq n\}$ and $\mathcal{H}^1 = \{N_i, 0 \leq i \leq n\}$. Define a class of diffeomorphisms $T_{\mathcal{H}, \mathcal{H}^1}$ by $f \in T_{\mathcal{H}, \mathcal{H}^1}$ if and only if $f(M_i) \subset \text{int } N_i$ for $0 \leq i \leq n$, and $f(D^i_j \times 0) \pitchfork (0 \times D^{n-i}_k)$ for $0 \leq i \leq n$ and all j, k where the $(D^i_j \times 0)$ are core discs of $\mathcal{H}$, and the $(0 \times D^{n-i}_k)$ are transverse discs of $\mathcal{H}^1$. By general position any diffeomorphism of M is isotopic to one in $T_{\mathcal{H},\mathcal{H}^1}$. Since the base points of $\mathcal{H}$ and $\mathcal{H}^1$ coincide we can construct an isotopy g_t $0 \leq t \leq 1$ such that $g_0 = Id_M$, $g = g_1 \in T_{\mathcal{H},\mathcal{H}^1}$ and

$g_t(x_0) = x_0$ $g_t(x_0') = x_0'$ for $0 \leq t \leq 1$. Therefore $g_*: \pi_1(M, x_0) \circlearrowleft$ is the identity. Since g is a homeomorphism $g_*: \tilde{\mathcal{C}}(M, \mathcal{H}) \to \tilde{\mathcal{C}}(M, \mathcal{H}^1)$ is certainly a chain equivalence. By Chapman's Theorem (Topological Invariance of Whitehead Torsion) [4], $\tau(g_*) = 0$. Therefore g_* is a simple chain equivalence. □

We will show that under certain assumptions a converse to (3.1) is also true. If $\mathcal{H}$ is a given handle decomposition of M and C^1 is an algebraic chain complex which is simple chain equivalent to $\tilde{\mathcal{C}}(M, \mathcal{H})$, then C^1 is realized by a handle decomposition $\mathcal{H}^1$ of M.

Observe that if $\mathcal{H}$ is a handle decomposition of a manifold M then the chain complex $\tilde{\mathcal{C}}(M, \mathcal{H})$ has a natural class of preferred bases. Furthermore, this is exactly the sense in which the geometry of $\mathcal{H}$ determines a basis for the chain complex. Therefore we will regard $\tilde{\mathcal{C}}(M, \mathcal{H})$ as a based $Z[\pi_1]$ complex.

3.2) <u>Definition</u> If $\mathcal{C}$ and $\mathcal{C}'$ are based $Z[\pi_1]$ complexes we will call a chain map $F: \mathcal{C} \to \mathcal{C}'$ a <u>based isomorphism</u> if for each i, $F_i: C_i \to C_i'$ is an isomorphism such that the image of a preferred basis of C_i is a preferred basis of C_i'. (†) Therefore a based isomorphism is a simple isomorphism but the converse need not be true. If there exists a based isomorphism $F: \mathcal{C} \to \mathcal{C}'$ we will write $\mathcal{C} \cong \mathcal{C}'$.

If $\mathcal{C} = (C_i, \partial_i)$ is a based $Z[\pi_1]$ complex and M is a manifold such that $\pi_1(M, x_0) \cong \pi_1$ we will say that a handle decomposition $\mathcal{H}$ of (M, x_0, x_0') <u>realizes</u> $\mathcal{C}$ if there exists a based isomorphism $F: \tilde{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$. Notice that in that case C_i is isomorphic to $\tilde{C}_i(M, \mathcal{H})$ and there exist natural choices of bases for $\tilde{\mathcal{C}}(M, \mathcal{H})$ so that the boundary matrices coincide (up to an identification of $\pi_1(M, x_0)$ with π_1).

3.3) We will need the notion of a based homotopy system which was also defined by Whitehead (S.H.T. #17). Let $\rho = (\rho_i, d_i)$ be a homotopy system; ρ has dimension n if $\rho_i = 0$ for $i > n$. ρ is <u>based</u> if each ρ_i has associated a class of <u>preferred bases</u> in the following sense. If $\{[a_1^i], \ldots, [a_r^i]\}$ is a preferred basis for ρ_i, $i \geq 2$, then any other preferred basis is of the form $\{\pm x_1[a^i_{\sigma(1)}], \ldots, \pm x_r[a^i_{\sigma(r)}]\}$, where σ is a permutation

(†) This terminology is non-standard. Otherwise we are following Whitehead's definitions in S.H.T. Cohen defines the class of preferred bases more loosely (an "(R, G) complex" [5, Chapter III #2]). This makes the theory more elegant, but his definition seems less useful for our problem of realization.

of $\{1, \ldots, r\}$ and the $x_j \in \rho_1$ when $i = 2$, and $x_j \in \bar{\rho}_1$ when $i \geq 3$. When $i = 1$ however we require that all the $x_i = 1$ in ρ_1. If ρ and ρ' are based homotopy systems we will say that a homomorphism $F: \rho \to \rho'$ is a based isomorphism if for each i the image of a preferred basis of ρ_i is a preferred basis of ρ_i'. (This is called simply an 'isomorphism' by Whitehead in S. H. T. # 17). If there exists a based isomorphism between ρ and ρ' we will write $\rho \approx \rho'$. Notice that if (ρ, d) is a based homotopy system then the induced chain complex $\mathcal{C}(\rho, d)$ is made a based $Z[\pi_1]$ complex by forgetting the stricter definition of preferred basis in dimension 1. If $\mathcal{H}$ is a handle decomposition of (M, x_0, x_0') then $\rho(M, \mathcal{H})$ and $\rho(M, \mathcal{H}')$ are based homotopy systems in a natural way. From now on we will assume all homotopy systems are based.

3.4) Definition If $\mathcal{C} = (C_i, \partial_i)$ is based $Z[\pi_1]$ complex, we will say that $\mathcal{C}$ admits homotopy groups if there exists a based homotopy system (ρ, d) and a based isomorphism $F: \mathcal{C}(\rho, d) \to \mathcal{C}$.

Thus a based chain complex $\mathcal{C}$ over $Z[\pi_1]$ admits homotopy groups provided there exists a presentation of π_1 compatible with

$(C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0)$. I have not been able to determine when this is true for an arbitrary $Z[\pi_1]$ complex.

We can now state the main result of this chapter.

3.5) <u>Theorem</u> Let M^n be a connected closed manifold of dimension $n \geq 9$. Let x_0, x_0' be chosen base points and $\mathcal{H}$ a handle decomposition of (M, x_0, x_0'). Suppose $\mathcal{C}$ is a based $Z[\pi]$ complex of dimension n such that both $\mathcal{C}$ and its dual complex admit homotopy groups. Then $\mathcal{C}$ is realized by a handle decomposition $\mathcal{H}^1$ of (M, x_0, x_0') if and only if there exists a simple chain equivalence $G : \breve{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$. □

The proof, with a few refinements, will occupy the rest of this chapter.

3.6) We now describe Whitehead's definition of simple equivalence for based homotopy systems, which will be our main algebraic tool.

Let $\rho = (\rho_i, d_i)$ be a homotopy system. We define a new homotopy system ρ' containing ρ as follows. Adjoin two new basis elements $[a_i^r]$, $[a_j^{r-1}]$, $r \geq 2$. If $r \geq 3$ let

$d_r'[a_i^r] = \epsilon\alpha[a_j^{r-1}] - [b^{r-1}]$ where $[b^{r-1}] \in \rho_{r-1}$ is arbitrary, $\epsilon = \pm 1$,

and $\alpha \in \bar{\rho}_1$ when $r \geq 4$ and $\alpha \in \rho_1$ when $r = 3$. When $r = 2$ let $d_2[a_i^2] = [x]^{-1}[a_j^1]^{\epsilon}[y]^{-1}$ where $[x], [y] \in \rho_1$ are arbitrary. It follows that $(\mathrm{inc}) : \rho \to \rho'$ is a chain equivalence. A homomorphism $F : \rho \to \rho'$ is an <u>elementary expansion of dimension r</u> if ρ, ρ' are as described and $F \simeq (\mathrm{inc})$. We will call $(\mathrm{inc}) : \rho \to \rho'$ the <u>standard</u>

elementary expansion. Define a homomorphism $k: \rho' \to \rho$ by $k_r[a_i^r] = 0$, $k_{r-1}[a_j^{r-1}] = \epsilon\alpha^{-1}[b^{r-1}]$ when $r \geq 3$, and $k/\rho =$ identity in all cases. When $r = 2$, let $k_1[a_j^1] = [xy]^{\epsilon}$. It follows that $k: \rho' \to \rho$ is a chain homotopy inverse for (inc.) A homomorphism $F: \rho' \to \rho$ is <u>elementary contraction of dimension</u> r if ρ', ρ are as above and $F \simeq k$. We will call k the <u>standard</u> elementary contraction. By an <u>elementary equivalence</u> we mean a homomorphism F of either type.

Let ρ, ρ' be two homotopy systems. We will call a homomorphism $F: \rho \to \rho'$ a <u>simple equivalence</u> if and only if there is a finite sequence of homotopy systems $\rho^1, \ldots, \rho^k$ and a sequence of elementary equivalences and based isomorphisms

$$\rho \xrightarrow{F^1} \rho^1 \xrightarrow{F^2} \rho^2 \to \cdots \xrightarrow{F^k} \rho^k \xrightarrow{F^{k+1}} \rho'$$

such that $F = F^{k+1} \circ \cdots \circ F^1$.

3.7) (Whitehead [26, Theorem 15]). A homomorphism $F: \rho \to \rho'$ is a simple equivalence if and only if the induced map downstairs $F_*: \mathcal{C}(\rho) \to \mathcal{C}(\rho')$ is a simple chain equivalence. □

3.8) Next we outline the proof of our theorem (3.5) which is somewhat complicated. Theorem (3.7) enables us to apply the results of Chapters One and Two. Under the same assumptions as in (3.5) let $\mathcal{H}$ be a handle decomposition of (M, x_0, x_0') and let $\mathcal{C}$ be a based $Z[\pi_1]$ complex. If $G: \tilde{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$ is a simple chain equivalence, and $\mathcal{C}$ admits homotopy groups, we obtain a diagram:

$$\begin{tikzcd} \rho(M, \mathcal{U}) \arrow[d] & & \rho \arrow[d] \\ \check{\mathcal{C}}(M, \mathcal{U}) \arrow[rr, "G"] & & \mathcal{C} \end{tikzcd}$$

By (2.4) there exists a lift of G, $F: \rho(M, \mathcal{U}) \to \rho$ which makes the diagram commute. By (2.6) and (3.7) F is a simple equivalence. Therefore F is equal to the composition of a finite sequence $\{F^i\}$ of elementary equivalences and based isomorphisms:

$$\rho(M, \mathcal{U}) \xrightarrow{F^1} \rho^1 \xrightarrow{F^2} \rho^2 \to \qquad \to \rho^k \xrightarrow{F^{k+1}} \rho$$

By the <u>dimension</u> of an elementary equivalence we mean the largest dimension of a basis element introduced or deleted. Suppose ρ^i has already been realized by a handle decomposition $\mathcal{U}^i$ of (M, x_0, x_0'). If $F^{i+1}: \rho(M, \mathcal{U}^i) \to \rho^{i+1}$ is an elementary equivalence of dimension $\leq n-3$ then ρ^{i+1} can be obtained from $\rho(M, \mathcal{U}^i)$ by standard operations on the handles. These are the same operations as in the proof of the s-cobordism theorem, but using our statement of the Whitney Lemma (1.1) we are able to realize elementary contractions down to the bottom dimensions, 2 and 3 (see Lemma (3.13) below).

In the top dimensions (n-1) and (n-2) we expect to use duality. If $F^{i+1} : \rho(M, \mathcal{U}^i) \to \rho^{i+1}$ is an elementary equivalence of dimension $\geq$ n-2, then downstairs on the chain level we have a simple chain equivalence

$$\begin{tikzcd} \rho(M, \mathcal{U}^i) \arrow[r, "F^{i+1}"] \arrow[d] & \rho^{i+1} \arrow[d] \\ \check{\mathcal{C}}(M, \mathcal{U}^i) \arrow[r, "(F^{i+1})_*"] & \mathcal{C}(\rho^{i+1}) \end{tikzcd}$$

As in Chapter Two we will write $\mathcal{D}$ for the appropriately defined duality functor (essentially $\mathrm{Hom}_{Z[\pi]}(\quad , Z[\pi])$). If we apply $\mathcal{D}$ to the diagram above we obtain a new diagram

$$\begin{tikzcd} \check{\mathcal{C}}(M, (\mathcal{U}^i)') \arrow[r, phantom, "\cong"] & \mathcal{D}(\check{\mathcal{C}}(M, \mathcal{U}^i)) & \mathcal{D}(\mathcal{C}(\rho^{i+1})) \arrow[l, "\mathcal{D}(F_*^{i+1})"'] \\ \rho(M, (\mathcal{U}^i)') \arrow[u] & & \end{tikzcd}$$

It is not hard to show that $\mathcal{D}(F^{i+1}{}_*)$ is again a simple chain equivalence (Lemma 3.15). However, I have not been able to show that the dual chain complex $\mathcal{D}(\mathcal{C}(\rho^{i+1}))$ admits homotopy groups. Since $\mathcal{D}(F_*^{i+1})$ may well involve changes in the bottom dimensions 1, 2 and 3 of the dual complexes, we can't proceed without this step. Instead we use the assumption that the final chain complex we are trying to realize, $\mathcal{C}$, has a dual chain complex $\mathcal{D}(\mathcal{C})$ which does admit homotopy groups. First we rearrange the original sequence $\{F^i\}$ so that all elementary equivalences of low dimension (≤ 5) occur before all those of high dimension ($\geq n-2$) (Lemma 3.12 Unfortunately, in order to prove this lemma, I need dimension $M \geq 9$.) We obtain a new sequence

$$\begin{tikzcd}
\rho(M, \mathcal{H}) \arrow[r, "F^1"] \arrow[d] & \rho^1 \arrow[r] & \cdots \arrow[r, "F^\ell"] & \rho^\ell \arrow[r, "F^{\ell+1}"] & \rho^{\ell+1} \arrow[r] & \cdots \arrow[r] & \rho^{m-1} \arrow[r, "F^m"] & \rho \arrow[d] \\
\tilde{\mathcal{C}}(M, \mathcal{H}) & & & & & & & \mathcal{C}
\end{tikzcd}$$

where $F = F^m \circ \cdots \circ F^1$, the $\{F^1, \ldots, F^\ell\}$ are based isomorphisms or elementary equivalences of dimension $\leq n-3$, and the $\{F^{\ell+1}, \ldots . F^m\}$ are based isomorphisms or elementary equivalences of dimension ≥ 6. The $\{F^1, \ldots, F^\ell\}$ can be realized without difficulty, so there exists a handle decomposition $\mathcal{H}^\ell$ of (M, x_0, x_0') such that

$$\begin{tikzcd}
\rho(M, \mathcal{H}^\ell) \arrow[r, phantom, "\approx"] \arrow[d] & \rho^\ell \arrow[d] \\
\tilde{\mathcal{C}}(M, \mathcal{H}^\ell) \arrow[r, phantom, "\cong"] & \mathcal{C}(\rho^\ell)
\end{tikzcd} .$$

Therefore $\mathcal{D}(\mathcal{C}(\rho^\ell))$ admits homotopy groups, namely $\rho(M, (\mathcal{H}^\ell)')$. Let $H = F^m \circ \ldots \circ F^{\ell+1}$.

$$\begin{tikzcd}
\rho(M, \mathcal{U}^{\ell}) \arrow[r, "H"] \arrow[d] & \rho \arrow[d] \\
\tilde{\mathcal{C}}(M, \mathcal{U}^{\ell}) \arrow[r, "H_*"] & \mathcal{C}(\rho)
\end{tikzcd}$$

H_* is a simple chain equivalence, and by assumption the dual complex of $\mathcal{C}$ admits homotopy groups. We will write these as $\rho' \longrightarrow \mathcal{D}(\mathcal{C})$. Applying $\mathcal{D}$ to the diagram above we obtain

$$\begin{tikzcd}
\rho' \arrow[d] & & \rho(M, (\mathcal{U}^{\ell})') \arrow[d] \\
\mathcal{D}(\mathcal{C}) \arrow[r, "\mathcal{D}(H_*)"] & \mathcal{D}(\tilde{\mathcal{C}}(M, \mathcal{U}^{\ell})) \arrow[r, phantom, "\cong"] & \tilde{\mathcal{C}}(M, (\mathcal{U}^{\ell})')
\end{tikzcd}$$

Again there exists a lift $H' : \rho' \to \rho(M, (\mathcal{U}^{\ell})')$ which is a simple equivalence. At this point we will abandon the remaining steps in our sequence $\{F^{\ell+1}, \ldots, F^{m}\}$ and find a new sequence of elementary equivalences for H' on the dual homotopy systems. A priori this might involve elementary equivalences of dimension (on the dual) $\geq n-3$. However, since H is a composition of elementary equivalence all of dimension ≥ 6, H' is essentially the identity in the top dimensions of the dual systems. We will show this implies we can find a sequence of elementary equivalences for H' all of dimension $\leq n-3$. These can then be realized to complete the proof of (3.5). $\square$

$$\begin{tikzcd} \rho(M, \mathcal{U}^{\ell}) \arrow[r, "H"] \arrow[d] & \rho \arrow[d] \\ \breve{\mathcal{C}}(M, \mathcal{U}^{\ell}) \arrow[r, "H_*"] & \mathcal{C}(\rho) \end{tikzcd}$$

H_* is a simple chain equivalence, and by assumption the dual complex of $\mathcal{C}$ admits homotopy groups. We will write these as $\rho' \longrightarrow \mathcal{D}(\mathcal{C})$. Applying $\mathcal{D}$ to the diagram above we obtain

$$\begin{tikzcd} \rho' \arrow[d] & & \rho(M, (\mathcal{U}^{\ell})') \arrow[d] \\ \mathcal{D}(\mathcal{C}) \arrow[r, "\mathcal{D}(H_*)"] & \mathcal{D}(\breve{\mathcal{C}}(M, \mathcal{U}^{\ell})) \cong & \breve{\mathcal{C}}(M, (\mathcal{U}^{\ell})') \end{tikzcd}$$

Again there exists a lift $H' : \rho' \to \rho(M, (\mathcal{U}^{\ell})')$ which is a simple equivalence. At this point we will abandon the remaining steps in our sequence $\{F^{\ell+1}, \ldots, F^{m}\}$ and find a new sequence of elementary equivalences for H' on the dual homotopy systems. A priori this might involve elementary equivalences of dimension (on the dual) $\geq n-3$. However, since H is a composition of elementary equivalence all of dimension ≥ 6, H' is essentially the identity in the top dimensions of the dual systems. We will show this implies we can find a sequence of elementary equivalences for H' all of dimension $\leq n-3$. These can then be realized to complete the proof of (3.5). $\square$

A priori a simple equivalence $\rho(M, \mathcal{H}) \to \rho$ could involve elementary expansions of dimension > dimension M. To control their dimensions we adapt a result of Wall. A <u>formal deformation</u> is the geometric analogue of a simple equivalence, defined for finite CW complexes. The dimension of a formal deformation is the largest dimension of cells introduced or collapsed. Wall proved the following theorem, improving Whitehead's previous bound by 1. (†).

<u>Theorem</u> (Wall [24]). Let $f: X \to Y$ be a simple homotopy equivalence of finite connected CW complexes. There exists a formal deformation $D: X \to Y$ such that $D \simeq f$ and dimension D $\leq \max\{$dimension X, dimension Y, $3\} + 1$. □

In order to apply Wall's result in our algebraic setting we use the following result of Whitehead.

(3.9) <u>Theorem</u>. (Whitehead [26; Theorem 17]). Let X be a finite CW complex, $\rho(X)$ its homotopy system, and ρ' an abstract homotopy system. If $F: \rho(X) \to \rho'$ is a chain homotopy equivalence, then ρ' can be realized by a complex X', in such a way that F is realized by a map $f: X \to X'$. □

(†) Actually Wall obtains a bound one dimension better, $\dim D \leq \max\{\dim X, \dim Y; 3\}$, by introducing a third type of elementary deformation, but we will not use this refinement.

(3.10) <u>Lemma.</u> Let $F : \rho(M, \mathcal{H}) \to \rho'$ be a simple equivalence, and suppose n = dimension M = dimension $\rho' \geq 3$. Then there exists a sequence of elementary equivalences and based isomorphisms $F^1, \ldots, F^k$ such that $F \doteq (F^k \circ \ldots \circ F^1)$ and for all i dimension $F^i \leq n + 1$.

<u>Proof</u>. We obtain a finite CW complex X from M by collapsing transverse discs of $\mathcal{H}$, so $\rho(X) \cong \rho(M, \mathcal{H})$. $F : \rho(X) \to \rho'$ is a chain homotopy equivalence so by (3.9) there exist a finite CW complex X' and a map $f: X \to X'$ such that $\rho(X') \approx \rho'$ and $f_\# = F$.

Since $f_\#$ is a simple equivalence, $f: X \to X'$ is a simple homotopy equivalence, and by Wall's result there exists a formal deformation $D: X \to X'$ such that $f \doteq D$ and dimension $D \leq n+1$. Therefore $D_\# \doteq f_\# = F$, and the elementary deformations in D induce a sequence of elementary equivalences of the same dimensions. □

Whitehead's definition of elementary contraction is quite restricted. If $[a_i^r]$ and $[a_j^{r-1}]$ are elements of preferred bases of ρ, an elementary contraction killing them only applies to ρ if $[a_i^r]$ never occurs in $d_{r+1}[a_s^{r+1}]$ for any s, and $[a_j^{r-1}]$ never occurs in $d_r[a_\ell^r]$ for $\ell \neq i$. Thus $[a_i^r]$ corresponds to a principal cell of a geometric complex and $[a_j^{r-1}]$ to a free face. We now remove this restriction. That is, we will allow an elementary contraction deleting a pair of basis elements $[a_i^r], [a_j^{r-1}]$ whenever the boundary $d_r: \rho_r \to \rho_{r-1}$ records algebraically ' one point of incidence ' of $[a_i^r]$ on $[a_j^{r-1}]$. Let $k: \rho \to \rho'$ be defined as before (3.6) but define the boundary operator d' of the new homotopy system ρ' by $d'_{r+1} = k_r \circ d_{r+1}$, $d'_r = k_{r-1} \circ d_r$. Otherwise $d'_i = d_i/\rho'_i$. It follows that $k: \rho \to \rho'$ is a chain equivalence.

3.11 Lemma. Let $k: \rho(M, \mathcal{H}) \to \rho'$ be an elementary contraction as redefined. Then k is a simple equivalence as originally defined.

Proof As in (3.10) we obtain a finite CW complex X from M by collapsing transverse discs of $\mathcal{H}$. $k: \rho(X) \to \rho'$ is a chain equivalence so by (3.9) there exists a finite CW complex X' such that $\rho(X') \approx \rho'$. The natural map realizing k $f: X \to X'$ is an "internal collapse", which is a simple homotopy equivalence [15, Appendix B Lemma 1]. Therefore $f_\# = k$ is a simple equivalence. □ (†)

3.12) Rearrangement Lemma

Suppose ρ and ρ' are homotopy systems of dimension ≥ 9 and $F: \rho \to \rho'$ is a simple equivalence. Then there exists a sequence of based isomorphisms and elementary equivalences

(†) It would be more satisfying to prove (3.10) and (3.11) purely algebraically but I do not see how to do this.

$$\rho \xrightarrow{F^1} \rho^1 \longrightarrow \quad \xrightarrow{F^\ell} \rho^\ell \xrightarrow{F^{\ell+1}} \rho^{\ell+1} \longrightarrow \quad \xrightarrow{F^m} \rho'$$

such that $F \simeq (F^m \circ_{\cdots} \circ F^1)$ and

dimension $F^i \leq n-3$ for $1 \leq i \leq \ell$,

dimension $F^i \geq 6$ for $\ell+1 \leq i \leq m$.

Proof. Suppose $F = (G^k \circ_{\cdots} \circ G^1)$ is an arbitrary sequence of elementary moves. We obtain the desired sequence by rearranging the G^i. Since we only claim $F \simeq (F^m \circ_{\cdots} \circ F^1)$ we can afford to replace each elementary equivalence among the G^i by the respective standard elementary equivalence (i.e. the inclusion or natural retraction) which we will also write as G^i. A composition of based isomorphisms is a based isomorphism, so we can assume these occur one at a time. The proof will follow by considering, case by case, when we can rearrange a sequence of three successive steps

$$\rho^i \xrightarrow{G^{i+1}} \rho^{i+1} \xrightarrow{G^{i+2}} \rho^{i+2} \xrightarrow{G^{i+3}} \rho^{i+3}$$

where G^{i+1} and G^{i+3} are elementary equivalences, G^{i+2} is a based isomorphism, and dimension G^{i+3} < dimension G^{i+1}.

<u>Case 1.</u> Suppose G^{i+1} is an elementary expansion of dimension r $(inc)_r : \rho^i \to \rho^{i+1}$ and G^{i+3} is an elementary expansion of dimension s $(inc)_s : \rho^{i+2} \to \rho^{i+3}$. Let ρ^{i+1} have adjoined basis elements $[a_h^r]$, $[a_j^{r-1}]$ where $d_r^{i+1}[a_h^r] = \epsilon\alpha[a_j^{r-1}] - [b^{r-1}]$, $\epsilon = \pm 1$, $\alpha \in \pi_1$ and $[b^{r-1}] \in \rho_{r-1}^i$, and let ρ^{i+3} have adjoined basis elements $[a_u^s]$, $[a_v^{s-1}]$ where $d_s^{i+3}[a_u^s] = \delta\beta[a_v^{s-1}] - [b^{s-1}]$, $\delta = \pm 1$, $\beta \in \pi_1$ and $[b^{s-1}] \in \rho_{s-1}^{i+2}$. If $s=2$ the form of $d_s^{i+3}[a_u^s]$ is different but the argument is essentially the same.

If $s \geq r$ we may not be able to reverse the order of G^{i+1} and G^{i+3}, since $G^{i+2}[a_h^r]$ or $G^{i+2}[a_j^{r-1}]$ might occur in the boundaries $d_s^{i+3}[a_u^s]$ or $d_{s-1}^{i+3}[a_r^{s-1}]$. If $s < r$ this cannot occur.

Define a new homotopy system ρ^{i+4} by adjoining $[a_u^s]$ and $[a_v^{s-1}]$ to ρ^i with $d_s^{i+4}[a_u^s] = \delta\beta[a_v^{s-1}] - (G^{i+2})^{-1}[b^{s-1}]$. Then $(inc)_s : \rho^i \to \rho^{i+4}$ is an elementary expansion of dimension s. Similarly, define ρ^{i+5} by adjoining $[a_h^r]$ and $[a_j^{r-1}]$ to ρ^{i+4} with $d_r^{i+5}[a_h^r] = d_r^{i+1}[a_h^r]$ so $inc_r : \rho^{i+4} \to \rho^{i+5}$ is an elementary expansion of dimension r. Finally we define a based isomorphism $\rho^{i+5} \to \rho^{i+3}$, essentially by extending the isomorphism G^{i+2} to be the identity on $[a_u^s]$ and $[a_v^{s-1}]$.

$$\begin{tikzcd}
& \rho^{i+1} \arrow[r, "\approx", "G^{i+2}"'] & \rho^{i+2} \arrow[dr, "(inc)_s", "G^{i+3}"'] & \\
\rho^{i} \arrow[ur, "(inc)_r", "G^{i+1}"'] \arrow[dr, "(inc)_s"'] & & & \rho^{i+3} \\
& \rho^{i+4} \arrow[r, "(inc)_r"] & \rho^{i+5} \arrow[ur, "\approx"'] &
\end{tikzcd}$$

It follows that the diagram commutes.

<u>Case 2.</u> Let G^{i+1} be an elementary expansion of dimension r $(inc)_r : \rho^i \to \rho^{i+1}$ and G^{i+3} an elementary contraction of dimension s $(k)_s : \rho^{i+2} \to \rho^{i+3}$. Suppose ρ^{i+1} has adjoined $[a^r_h]$ and $[a^{r-1}_j]$, and $(k)_s$ deletes $[a^s_u]$ and $[a^{s-1}_v]$. If $r \geq s+2$ then the two pairs are disjoint and we can rearrange the three steps. Let $(G^{i+2})^{-1}[a^s_u] = \epsilon\alpha[a^s_w]$ and $(G^{i+2})^{-1}[a^{s-1}_v] = \delta\beta\, [a^{s-1}_z]$. Provided $s > 2$ $d^i_s\,[a^s_w]$ has the same form as $d^{i+2}_s\,[a^s_u]$ and we can apply an elementary contraction killing $[a^s_w]$ and $[a^{s-1}_z]$ directly to ρ^i. If $s = 2$ we might first need a based isomorphism, e.g. if $d^i_2\,[a^2_w] = [a^1_z] \ldots [a^1_z] \ldots [a^1_z]^{-1}$. In either case we can then apply an elementary expansion adding $[a^r_h]$ and $[a^{r-1}_v]$, and finally a based isomorphism which is essentially the restriction of G^{i+2}. We define intermediate steps ρ^{i+4}, ρ^{i+5} as in case 1, and obtain a commutative diagram

$$\begin{tikzcd}
 & \rho^{i+1} \arrow[r, "\approx", "G^{i+2}"'] & \rho^{i+2} \arrow[dr, "(k)_s", "G^{i+3}"'] & \\
\rho^{i} \arrow[ur, "(inc)_r", "G^{i+1}"'] \arrow[dr, "(k)_s"'] & & & \rho^{i+3} \\
 & \rho^{i+4} \arrow[r, "(inc)_r"] & \rho^{i+5} \arrow[ur, "\approx"'] &
\end{tikzcd}$$

Therefore in this case we can rearrange the three steps provided $(s+1) < r$.

<u>Case 3.</u> Let G^{i+1} be an elementary contraction of dimension r $(k)_r: \rho^i \to \rho^{i+1}$, G^{i+3} an elementary contraction of dimension s $(k)_s: \rho^{i+2} \to \rho^{i+3}$. Let $[a_h^r]$ $[a_j^{r-1}]$ and $[a_u^s]$ $[a_v^{s-1}]$ be the pairs deleted by $(k)_r$ and $(k)_s$ respectively. Let $(G^{i+2})^{-1}[a_u^s] = \epsilon\alpha[a_w^s]$ and $(G^{i+2})^{-1}[a_v^{s-1}] = \delta\beta[a_z^{s-1}]$. Notice that either $[a_w^s]$ or $[a_v^{s-1}]$ could appear in $d_r^i[a_h^r]$ or $d_{r-1}^i[a_j^{r-1}]$ so as originally defined an elementary contraction killing $[a_w^s]$ and $[a_v^{s-1}]$ might not apply to ρ^i. However the generalized definition (3.11) avoids this problem. If $s = 2$ again we may need a preliminary based isomorphism.

<u>Case 4</u> Finally suppose G^{i+1} is an elementary contraction and G^{i+3} an elementary expansion. We can reverse the order of such a pair regardless of dimension.

We now prove the lemma by induction. Let $F = G^k \circ \ldots \circ G^1$. Let $G^{i_1} \ldots G^{i_p}$ be all the elementary equivalences of dimension $\geq n-2$. If there are no elementary equivalences of dimension ≤ 5 after G^{i_p} go on to $G^{i_{p-1}}$. Otherwise let G^{i_p+t} be the first. (Of course we need $n-2 > 5$). Using cases 1-4 we commute successive pairs of elementary equivalences to bring G^{i_p+t} forward past G^{i_p}. Case 2, $(k)_s \circ (inc)_r \Rightarrow (inc)_r \circ (k)_s$ requires $r > s+1$, so if F^{i_p+t} is an elementary contraction of dimension $r \leq 5$, we could be blocked by an elementary expansion of dimension $(r+1)$ between F^{i_p} and F^{i_p+t}, or by a chain of such elementary expansions. In that case, start at the other end of the chain and move these elementary expansions, all of which have dimension $r+1 < n-2$ forward past F^{i_p}. However we do need

$\dim F^{i_p} \geq \dim (F^{i_p+t}) + 2$ i.e. $n-2 \geq 5+2$ or $n \geq 9$. □

If dimension $M = n \geq 6$ and $F : \rho(M, \mathscr{U}) \to \rho^1$ is an elementary equivalence of dimension $\leq n-3$, then we can realize ρ^1 by the well known processes of cancelling and introducing complementary handles ([15, 6.4 - 6.6], [8, Lemma 3], [10, Chapter 5]. Since F is defined on the homotopy system $\rho(M, \mathscr{U})$ rather than on the chain complex $\tilde{\mathcal{C}}(M, \mathscr{U})$ this process works down to dimension $(F) = 2$.

We verify that it has the desired effect on the boundary operators as well.

3.13) <u>Lemma.</u> Let M^n be a connected closed manifold, x_0, x_0' base points, and suppose $n \geq 6$. Let $\mathcal{H}$ be a handle decomposition of (M, x_0, x_0') and $F: \rho(M, \mathcal{H}) \to \rho^1$ an elementary equivalence of dimension $\leq n-3$. Then there exists a handle decomposition $\mathcal{H}^1$ of (M, x_0, x_0') and a based isomorphism $G: \rho(M, \mathcal{H}^1) \to \rho^1$. If F is monic then there also exists a diffeomorphism $g: M \to M$ isotopic to the identity, such that $g \in T_{\mathcal{H}, \mathcal{H}^1}$ and $g_\#: \rho(M, \mathcal{H}) \to \rho(M, \mathcal{H}^1)$ realizes F, i.e. $G \circ g_\# = F$.

$$\begin{tikzcd}
\rho(M, \mathcal{H}) \arrow[r, "g_\#"] \arrow[dr, "F"'] & \rho(M, \mathcal{H}^1) \arrow[d, "G"] \\
& \rho^1
\end{tikzcd}$$

<u>Proof.</u> Suppose first that F is an elementary contraction and $k: \rho(M, \mathcal{H}) \to \rho^1$ is the standard elementary contraction. Let $[\phi_i^r]$, $[\phi_j^{r-1}]$ be the generators deleted in ρ^1. If $r \geq 4$ then $d_r[\phi_i^r] = \epsilon\alpha[\phi_j^{r-1}] - [b^{r-1}]$ where $\epsilon = \pm 1$, $\alpha \in \pi_1$ and $[b^{r-1}] = \sum_{\ell \neq j} \gamma_\ell [\phi_\ell^{r-1}]$, $\gamma_\ell \in Z[\pi_1]$. If $r = 3$ $d_3[\phi_i^3] = A + \epsilon\alpha[\phi_j^2] + B$ where $\alpha \in \rho_1$, $A, B \in \rho_2$ and $[\phi_j^2]$ does not occur in A or B. When r=2 we

have $d_2[\phi_i^2] = [x]^{-1}[\phi_j^1]^{\epsilon}[y]^{-1}$ where $\epsilon = \pm 1$ and $[x], [y] \in \rho_1^1$. Therefore it follows from the Cancelling Lemma in Chapter One (1.1) that, provided $r \leq n-3$, the attaching map $\phi_i^r : (S_i^{r-1} \times 0) \to \partial M_{r-1}$ is isotopic to an embedding $(\phi_i^r)^1$ such that $(\phi_i^r)^1(S_i^{r-1} \times 0) \pitchfork (0 \times S_j^{n-r})$ consists of a single point. By the proof of (1.1) we can assume that the isotopy avoids the attaching spheres of all the other (r)- handles. Therefore we obtain a new handle decomposition of M, with attaching map $(\phi_i^r)^1$, such that the homotopy system $\rho(M, \mathscr{U})$ is unchanged. We will assume this is done and still call the handle decomposition $\mathscr{U}$.

Think of the handle decomposition as arising from a Morse function as in Milnor's book [10]. Let $h : M \to [0,1]$ be a Morse function, and η a gradient - like vector field giving rise to $\mathscr{U}$. Let ϕ_i^r, ϕ_j^{r-1} correspond to critical points p, q of h. In these terms $\phi_i^r(S_i^{r-1} \times 0) \pitchfork (0 \times S_j^{n-r}) = 1$ point means there is exactly one trajectory, T, of η from p to q. Therefore we can apply the following cancelling lemma from the proof of the h-cobordism theorem. (We paraphrase Milnor's statement).

<u>Theorem.</u> Let $h : M \to [0,1]$ be a Morse function, η a gradient - like vector field for h. Suppose p, q are critical points of h of successive index such that there is exactly one trajectory T from p to q. Then it is possible to alter η in a

small neighborhood N(T) to obtain a new gradient-like vector field η^1 which is never zero on N(T). η^1 is a gradient-like vector field for a Morse function h^1 without critical points in N(T), which equals h outside N(T). □

Let $\mathcal{H}^1$ be the handle decomposition associated with (h^1, η^1). Clearly $\rho(M, \mathcal{H}^1)$ has the correct number of generators in each dimension. Since η is unaltered outside N(T) $d_i^{\mathcal{H}^1} = d_i^{\mathcal{H}} / \rho_i(M, \mathcal{H}^1)$ except for $i = r+1, r$. Let ϕ_ℓ^{r+1} be a handle of $\mathcal{H}$ corresponding to a critical point s of the Morse function h, and let p and q be as above. Under the altered gradient - like vector field η', trajectories that used to flow from s to p now continue down to M_{r-1}. Therefore $d_{r+1}^{\mathcal{H}^1}[\phi_\ell^{r+1}] = k_r \circ d_{r+1}[\phi_\ell^{r+1}] = d_{r+1}^{\rho^1}[\phi_\ell^{r+1}]$. Thus $d_{r+1}^{\mathcal{H}^1}$ has the required form.

Next we show that $d_r^{\mathcal{H}^1}$ has the required form, $d_r^{\mathcal{H}^1} = k_{r-1} \circ d_r = d_r^{\rho^1}$. Suppose first that $r \geq 4$ so $(r-1) \geq 3$. We have $d_r[\phi_i^r] = \epsilon\alpha[\phi_j^{r-1}] - [b^{r-1}]$ and $k_{r-1}[\phi_j^{r-1}] = \epsilon\alpha^{-1}[b^{r-1}]$, where $[b^{r-1}] = \sum_{m \neq j} \gamma_m[\phi_m^{r-1}]$ and the γ_m are $Z[\pi_1]$ incidence numbers. Let $[\phi^r_\ell]\ \ell \neq i$ be an (r) - handle of $\mathcal{H}$, $[\phi^{r-1}_m]\ m \neq j$ an (r-1) - handle of $\mathcal{H}$, and suppose they correspond respectively to critical points t, s of the Morse function $h: M \to [0,1]$. When η is altered one new trajectory of η' from t to s is created for each pair consisting of a trajectory of η from t to q and a trajectory of η from p to s. Comparing base paths and orientations it follows that $d^{\mathcal{H}^1}_r [\phi^r_\ell] = k_{r-1} \circ d_r [\phi^r_\ell]$ as desired. When $r = 3$ $d_3[\phi^3_\ell]$ is defined with operators in ρ_1 but the argument adapts. Suppose $r = 2$. Then we have $d_2[\phi^2_i] = [x]^{-1}[\phi^1_j]^{\epsilon}[y]^{-1}$. Suppose the attaching map of ϕ^2_ℓ hits ϕ^1_j at a point v with orientation δ. Then when η is altered $\phi^2_\ell(S^1_\ell \times 0)$ will be isotoped acorss what was $(D^2_i \times 0)$ to hit $[xy]^{\epsilon\delta}$. Therefore

$d_2^{\mathcal{H}^1}[\phi^2_\ell] = k_1 d_2[\phi^2_\ell]$. It follows that $\rho(M, \mathcal{H}^1)$ realizes ρ^1, i.e. there exists a based isomorphism $G: \rho(M, \mathcal{H}^1) \to \rho^1$.

We now show how to obtain a diffeomorphism g realizing $k: \rho(M, \mathcal{H}) \to \rho^1$. If we deform Id_M along the flow lines of the altered gradient-like vector field η' we obtain a diffeomorphism $g \in T_{\mathcal{H}, \mathcal{H}^1}$ isotopic to the identity. During the isotopy the image of $(D^r_i \times 0)$ flows down to hit $int(M^1_{r-1})$ (the (r-1) "skeleton" of $\mathcal{H}^1$). Therefore $g_{\# r}[\phi^r_i] = 0$. At the same time the image of $(D^{r-1}_j \times 0)$ flows across what was $(D^r_i \times 0)$ to hit the other (r-1) handles in the boundary of $(D_i^r \times 0)$. Now in $\mathcal{H}$ we had

$[(\Gamma_i^r)(\Gamma_j^{r-1})^{-1}] = \alpha$. Therefore after the image of $(D_j^{r-1} \times 0)$

flows across $(D_i^r \times 0)$ we obtain

$$g_{\# r-1}[\phi_j^{r-1}] = \epsilon\alpha^{-1}[b^{r-1}] = k_{r-1}[\phi_j^{r-1}] \; .$$

Thus $g_\#$ realizes $k \colon \rho(M, \mathcal{H}) \to \rho^1$, i.e. $G \circ g_\# = k$.

$$\begin{tikzcd}
\rho(M, \mathcal{H}) \arrow[r, "g_\#"] \arrow[dr, "k"'] & \rho(M, \mathcal{H}^1) \arrow[d, "G"] \\
& \rho^1
\end{tikzcd}$$

We were given $F \colon \rho(M, \mathcal{H}) \to \rho^1$ an elementary contraction so $F \doteq k$. If F is monic by the results of Chapter Two we obtain a diffeomorphism $g' \in T_{\mathcal{H}, \mathcal{H}^1}$ isotopic to the identity such that $(g')_\#$ realizes F.

This completes the proof when $F \colon \rho(M, \mathcal{H}) \to \rho^1$ is an elementary contraction. When F is an elementary expansion we reverse the argument and show that an n-disc attached to ∂M_{r-2} can be regarded as a complementary pair of handles of index $r, r-1$. The proof is essentially the same as [15 , Lemma 6.6]. Since we haven't modified the definition of elementary expansions we must build the pair ϕ_i^r, ϕ_j^{r-1} so that $[\phi_i^r]$ never appears in

$d^{\mathcal{H}^1}_{r+1}([\phi^{r+1}_s])$ for all s and $[\phi^{r-1}_j]$ never appears in $d^{\mathcal{H}^1}_r[\phi^r_\ell]$ for $\ell \neq i$. Provided $r \leq n-3$ this can be done by building the pair in the union of the (n) and (n-1)-handles of $\mathcal{H}$. $\square$

If $\mathcal{H}$ is a handle decomposition and ϕ^r_i, ϕ^{r-1}_j are handles such that $r \geq 4$ and the $Z[\pi_1]$ incidence number $\partial^r_{ij} = \pm\alpha$, $\alpha \in \pi_1$, but $r > n-3$, then we are no longer guaranteed that the attaching map of ϕ^r_i can be isotoped to realize one point of intersection in $\phi^r_i(S^{r-1}_i \times 0) \pitchfork (0 \times S^{n-r}_j)$. Therefore when $F: \rho(M, \mathcal{H}) \to \rho^1$ is an elementary contraction of dimension $> n-3$ the proof of (3.13) breaks down. We could still realize F if there was an 'equivalent' elementary contraction defined on the homotopy system of the dual decomposition $\rho(M, \mathcal{H}')$, provided that contraction had dimension $\leq n-3$. However when $r > n-3$ the boundary of the dual handle $d'_{n-r+1}[\phi'^{\,n-r+1}_j]$ is more sensitive than $\partial^r(\phi^r_i)$. Therefore the dual of F may not be an elementary contraction. We will have to do a little more work to make Poincare' duality work for us. We first show that if $G: \mathcal{C} \to \mathcal{D}$ is a simple chain equivalence so is its suitably defined dual.

Recall from Lemma (3.1) that if $\mathcal{H}$, $\mathcal{H}^1$ are handle decompositions of (M, x_0, x_0') then the simple chain equivalence $g_*: \breve{\mathcal{C}}(M, \mathcal{H}) \to \breve{\mathcal{C}}(M, \mathcal{H}^1)$ was associated with the identity on $\pi_1(M, x_0)$. It also follows from the proof of (3.1) that if ϕ^n is the (n) - handle of $\mathcal{U}$, and Ψ^n that of $\mathcal{U}^1$, then $g_*(\phi^n) = +\Psi^n$ i.e. in the notation of (2.11) $\epsilon_n \alpha_n = +e$. For the rest of the chapter we restrict attention to chain maps having these properties.

Define a category of chain complexes $\mathcal{C}(\pi_1)$ as follows. Objects are based $Z[\pi_1]$ complexes

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \to \cdots\cdots \to C_0 \longrightarrow 0.$$

Each object will be assigned a formal dimension n such that $C_i = 0$ for $i > n$ but n need not be the least such integer. Morphisms are chain maps $F: \mathcal{C} \to \mathcal{D}$ associated with the identity on π_1.

Define a functor $\mathcal{D} : \mathcal{C}(\pi_1) \circlearrowleft$ as follows. Let $\mathcal{C}$ be a complex in $\mathcal{C}(\pi_1)$ with dimension $\mathcal{C} = n$. $\mathcal{D}(\mathcal{C})_{n-k}$ will be a free $Z[\pi_1]$ complex of the same rank as C_k. Choose a family of preferred bases for the C_k ; $[a_1^k]$, ..., $[a_r^k]$, and let formal elements $[a_1^{n-k}]'$ . . . $[a_r^{n-k}]'$ generate $\mathcal{D}(\mathcal{C})_{n-k}$ as a free $Z[\pi_1]$ - module, and determine its class of preferred bases. If the matrix (∂_{ij}^k) represents $\partial_k : C_k \to C_{k-1}$ with respect to the chosen family of bases for $\mathcal{C}$, then the matrix representing the dual boundary $\partial^{\mathcal{D}(\mathcal{C})} : \mathcal{D}(\mathcal{C})_{n-k+1} \to \mathcal{D}(\mathcal{C})_{n+k}$ with respect to the dual bases has the same form as in (2.10), but we forget the isomorphism $\Phi_\gamma : \pi_1(M, x_0) \to \pi_1(M, x_0')$. That is,

$$(\partial^{\mathcal{D}(\mathcal{C})})_{ji}^{n-k+1} = (-1)^k \ ((\partial^{\mathcal{C}})_{ij}^{k})^* .$$

If a chain map $F : \mathcal{C} \to \mathcal{D}$ is represented by matrices (a_{ij}^k) then let

$$\mathcal{D}(F)_{ji}^{n-k} = (a_{ji}^k)^*$$

The following proposition can be proved in the same way as (2.10) - (2.13).

3.14) <u>Proposition</u>. $\mathcal{D} : \mathcal{C}(\pi_1) \circlearrowleft$ is a contravariant functor. If $F, G : \mathcal{C} \to \mathcal{D}$ are chain homotopic so are $\mathcal{D}(F), \mathcal{D}(G) : \mathcal{D}(\mathcal{D}) \to \mathcal{D}(\mathcal{C})$. Therefore if $F : \mathcal{C} \to \mathcal{D}$ is a chain equivalence so is $\mathcal{D}(F) : \mathcal{D}(\mathcal{D}) \to \mathcal{D}(\mathcal{C})$.

<u>Remark</u> It is no longer possible to ignore the fact that the functor "$\mathcal{D}$" is essentially $\mathrm{Hom}_{Z[\pi_1]}(\quad , Z[\pi_1])$. If C is a left $Z[\pi_1]$ - module we make $\mathrm{Hom}_{Z[\pi_1]}(C, Z[\pi_1])$ a left $Z[\pi_1]$ - module in the following way [13]. For $\gamma \in Z[\pi_1]$, $\lambda \in \mathrm{Hom}_{Z[\pi_1]}(C, Z[\pi_1])$ and $c \in C$ define $\gamma\lambda$ by $\gamma\lambda(c) = \lambda(c)(\gamma)^*$. If $\mathcal{C} \in \mathcal{C}(\pi_1)$ then up to the sign factor, $(-1)^k$,

$\mathcal{D}(\mathbb{C})$ is canonically isomorphic to $\mathrm{Hom}_{Z[\pi_1]}(\mathbb{C}, Z[\pi_1])$. In Chapter Two it seemed necessary to keep the information on base points and base paths which obscured this. For the sake of internal consistency we will keep the notation "$\mathcal{D}$".

3.15) <u>Lemma.</u> If $G: \mathbb{C} \to \mathbb{D}$ is a simple chain equivalence in $\mathcal{C}(\pi_1)$ so is $\mathcal{D}(G): \mathcal{D}(\mathbb{D}) \to \mathcal{D}(\mathbb{C})$.

<u>Proof</u> By definition there exist collapsible complexes $\mathbb{B}$, $\mathbb{B}'$ and a simple isomorphism $F: \mathbb{C} \oplus \mathbb{B} \to \mathbb{D} \oplus \mathbb{B}'$ such that $G \simeq k \circ F \circ inc$ where $k: \mathbb{D} \oplus \mathbb{B}' \to \mathbb{D}$ is some retraction and $inc: \mathbb{C} \to \mathbb{C} \oplus \mathbb{B}$ is the inclusion.

$$\begin{tikzcd}
\mathbb{C} \oplus \mathbb{B} \arrow[r, "F"] & \mathbb{D} \oplus \mathbb{B}' \arrow[d, "k"] \\
\mathbb{C} \arrow[u, "inc"] \arrow[r, "G"] & \mathbb{D}
\end{tikzcd}$$

Regard all these complexes as having the same dimension and apply the functor $\mathcal{D}$.

$$\begin{tikzcd}
\mathcal{D}(\mathbb{C} \oplus \mathbb{B}) \arrow[d, "\mathcal{D}(inc)"] & \mathcal{D}(\mathbb{D} \oplus \mathbb{B}') \arrow[l, "\mathcal{D}(F)"'] \\
\mathcal{D}(\mathbb{C}) & \mathcal{D}(\mathbb{D}) \arrow[l, "\mathcal{D}(G)"] \arrow[u, "\mathcal{D}(k)"']
\end{tikzcd}$$

Now $\mathcal{D}(\mathbb{D}\oplus\mathbb{B}') \cong \mathcal{D}(\mathbb{D}) \oplus \mathcal{D}(\mathbb{B}')$, $\mathcal{D}(\mathcal{C}\oplus\mathbb{B}) \cong \mathcal{D}(\mathcal{C}) \oplus \mathcal{D}(\mathbb{B})$ and $\mathcal{D}(\mathbb{B})$, $\mathcal{D}(\mathbb{B}')$ are still collapsible. Let $<F_k>$ be the matrix representing F_k with respect to some choice of preferred bases, so $\tau(<F_k>) = 0$. Then with respect to the dual preferred bases $<\mathcal{D}(F)_{n-k}> = <F_k>^{*,\text{transpose}}$ so $\tau<\mathcal{D}(F)_{n-k}> = 0$ as well [12, Remark p. 398]. Therefore $\mathcal{D}(F)$ is a simple isomorphism. $\mathcal{D}(\text{inc}): \mathcal{D}(\mathcal{C}\oplus\mathbb{B}) \to \mathcal{D}(\mathcal{C})$ is a retraction. $\mathcal{D}(k)$ may not be the inclusion but since $\mathcal{D}(\mathbb{B}')$ is collapsible $1_{\mathcal{D}(\mathbb{B}')} \simeq 0$, so

$\mathcal{D}(k) \simeq (1_{\mathcal{D}(\mathbb{D})} \oplus 0_{\mathcal{D}(\mathbb{B}')}) \circ \mathcal{D}(k) = \text{inc}: \mathcal{D}(\mathbb{D}) \to \mathcal{D}(\mathbb{D}) \oplus \mathcal{D}(\mathbb{B}')$.

Therefore $\mathcal{D}(G) \simeq \mathcal{D}(\text{inc}) \circ \mathcal{D}(F) \circ (\text{inc})$ and the lemma is proved.

$\square$

The next two lemmas enable us to control the dimension of the elementary equivalences required when we dualize an elementary equivalence of high dimension (> n-3).

If $\mathcal{C}$ is a based $Z[\pi_1]$ chain complex of dimension $n > m$, by $\mathcal{C}/m$ we mean the chain complex obtained by truncating $\mathcal{C}$ at dimension m.

$$0 \longrightarrow C_m \xrightarrow{\partial} C_{m-1} \cdots\cdots\cdots \longrightarrow C_0 \longrightarrow 0$$

If $F: \mathcal{C} \to \mathcal{D}$ is a chain map so is $F/: \mathcal{C}/_m \to \mathcal{D}/m$.

3.16) Lemma Let $F: \mathcal{C} \to \mathcal{D}$ be a chain equivalence of based $Z[\pi_1]$ complexes of dimension n. If F_i is in fact an isomorphism for $i > m$ then $F/ : \mathcal{C}/m \to \mathcal{D}/m$ is also a chain equivalence. If F is actually a simple chain equivalence, and $\tau(F_i) = 0$ for $i > m$ then $F/ : \mathcal{C}/m \to \mathcal{D}/m$ is a simple chain equivalence.

Proof. By Whitehead's Theorem $F/ : \mathcal{C}/m \to \mathcal{D}/m$ will be a chain equivalence if and only if the induced map on homology $(F/\ )_{*k} : H_k(\mathcal{C}/m) \to H_k(\mathcal{D}/m)$ is an isomorphism for all k. For $k < m$ this is immediate. Suppose that F_i is an isomorphism for $i > m$; we show $(F/\ )_{*m} : H_m(\mathcal{C}/m) \to H_m(\mathcal{D}/m)$ is also an isomorphism. Now $H_m(\mathcal{C}/m) = \text{kernel } \partial_m^{\mathcal{C}}$ and $H_m(\mathcal{D}/m) = \text{kernel } \partial_m^{\mathcal{D}}$. Let $d \in \text{kernel } \partial_m^{\mathcal{D}}$.

$[d] \in H_m(\mathcal{D})$ so $\exists\ c \in \text{kernel } \partial_m^{\mathcal{C}}$, $F_*[c] = [d]$

i.e. $\exists\ d' \in D_{m+1}$, $F(c) = d + \partial_{m+1}^{\mathcal{D}}(d')$. Since F_{m+1} is an isomorphism $\exists\ c' \in C_{m+1}$, $F(c') = d'$. Now $\partial_m^{\mathcal{C}}(c - \partial c') = 0$ and $F(c - \partial c') = d + \partial_{m+1}^{\mathcal{D}}(d') - \partial_{m+1}^{\mathcal{D}}(d') = d$. Therefore

$$(F/\ )_* : H_m(\mathcal{C}/m) \longrightarrow H_m(\mathcal{D}/m) \text{ is onto.}$$

Let $c_1, c_2 \in \text{kernel } \partial_m^{\mathcal{C}}$ and suppose $F(c_1) = F(c_2)$. We also have $[c_1], [c_2] \in H_m(\mathcal{C})$ so $F_*[c_1] = F_*[c_2]$. Since F_{*m} is an isomorphism (i.e. on the homology of the complete complexes) $\exists\ c' \in C_{m+1},\ \partial^{\mathcal{C}}_{m+1}(c') = c_1 - c_2$. Let $F(c') = d'$, $\partial^{\mathbb{D}}_{m+1}(d') = 0$ so $[d'] \subset H_{m+1}(\mathbb{D})$ and $\exists\ c'' \in$ kernel $\partial^{\mathcal{C}}_{m+1}$, $F_*[c''] = [d']$. Therefore there must be $d'' \in D_{m+2}$ $F(c'') = d' + \partial^{\mathbb{D}}_{m+2}(d'')$. Since F_{m+2} is also assumed to be an isomorphism there exists $c''' \in C_{m+2}$ such that $F_{m+2}(c''') = d''$. Consider $(c'' - \partial c''') \in C_{m+1}$. $\partial^{\mathcal{C}}_{m+1}(c'' - \partial c''') = 0$ and $F_{m+1}(c'' - \partial c''') = F_{m+1}(c'') - \partial_{m+2} F_{m+1}(c''')$ $= d' + \partial d'' - \partial d'' - \partial d'' = d'$. But $F_{m+1}(c') = d'$ and F_{m+1} is one-one. Therefore $c' = (c'' - \partial c''')$ and $c_1 - c_2 = \partial^{\mathcal{C}}_{m+1}(c') = 0$. Hence $(F/\ \ )_{*m}$ is one-one and hence an isomorphism. This proves the first assertion in the lemma.

Now assume that $F: \mathcal{C} \to \mathbb{D}$ is a simple chain equivalence, and $\tau(F_i) = 0$ for $i > m$. Let $E = E(F)$ be the algebraic mapping cone, $E_k = C_{k-1} \oplus D_k$, $E_{k-1} = C_{k-2} \oplus D_{k-1}$ with boundary ∂^E_k defined by

$$\partial^E_k(c, d) = (-\partial^{\mathcal{C}}_{k-1}(c),\ F_{k-1}(c) + \partial^{\mathbb{D}}_k(d)).$$

Let $\mathcal{C}^{\backslash m+1}$ be the chain complex truncated at the bottom

$$0 \longrightarrow C_n \longrightarrow \cdot\ \cdot\ \cdot\ \cdot\ \cdot \longrightarrow C_{m+1} \longrightarrow 0.$$

It is easy to see that $F^{\backslash m+1}: \mathcal{C}^{\backslash m+1} \to \mathbb{D}^{\backslash m+1}$ is a chain equivalence. Therefore the algebraic mapping cones $E(F^{\backslash m+1})$ $E(F/_m)$ are defined and we get a based short exact sequence:

$$0 \longrightarrow E(F/_m) \longrightarrow E(F) \longrightarrow E(F\backslash^{m+1}) \longrightarrow 0$$

(The sequence is based means that for each i a preferred basis of $E(F/m)_i$ is mapped one-one into a subset of a preferred basis of $E(F)_i$, and the complementary subset of the preferred basis of $E(F)_i$ is mapped one-one onto a preferred basis of $E(F\backslash^{m+1})_i$).

Except in dimensions $m+2$, $m+1$, m this is trivially true. In these dimensions we obtain the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & E(F/_m) \arrow[r] & E(F) \arrow[r] \arrow[d] & E(F\backslash^{m+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_{m+1} \oplus D_{m+2} \arrow[r, "1"] \arrow[d] & C_{m+1} \oplus D_{m+2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_m \oplus 0 \arrow[r, "inc"] \arrow[d] & C_m \oplus D_{m+1} \arrow[r, "0 \times 1"] \arrow[d] & 0 \oplus D_{m+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{m-1} \oplus D_m \arrow[r, "1"] \arrow[d] & C_{m-1} \oplus D_m \arrow[r, "0"] \arrow[d] & 0 \arrow[r] & 0 \\
 & {} & {} & &
\end{tikzcd}$$

It is easy to see that all squares commute and the sequence is based exact. It follows [24, p. 344 Condition P3] that

$$\tau(E(F)) = \tau(E(F/_m)) + \tau(E(F\backslash^{m+1}) \;.$$

Since F is a simple chain equivalence $\tau(F) = \tau(E(F)) = 0$, and since $\tau(F_i) = 0$ for $i > m$, $\tau(E(F\backslash^{m+1})) = 0$ also. Therefore $\tau(E(F/_m)) = 0$ and $F/_m : \mathcal{C}/_m \to \mathcal{D}/m$ is a simple chain equivalence.

□

3.17) Lemma. Let $\rho = \rho(M, \mathcal{U})$ be a realized homotopy system, ρ' another homotopy system, both of dimension n and $F: \rho \to \rho'$ a simple equivalence. Suppose $n > m \geq 3$ and, with respect to some chosen families of preferred bases, for all $i > m$ the matrix representing $F_i : \rho_i \to \rho'_i$ is the identity. Then there exists a sequence of elementary equivalences and based isomorphisms

$$\rho \xrightarrow{F^1} \rho^1 \longrightarrow \cdot \cdot \cdot \cdot \;\; \rho^q \xrightarrow{F^{q+1}} \rho'$$

such that $F = (F^{q+1} \circ \cdot \cdot \cdot \circ F^1)$ and dimension $F^j \leq m+1$ for $1 \leq j \leq q+1$.

Proof. By (3.16) $F/_m : \rho/_m \to \rho'/_m$ is also a simple equivalence. By (3.10) there exists a sequence of elementary equivalences and based isomorphisms, all of dimension $\leq m+1$, whose composition equals $F/ : \rho/_m \to \rho'/_m$.

$$\rho/_m \xrightarrow{G^1} \rho^1 \xrightarrow{G^2} \rho^2 \to \cdots\cdots \to \rho^q \xrightarrow{G^{q+1}} \rho'/_m$$

The proof follows by extending the ρ^j and G^j. We start at the lower left hand corner and work up.

$$\begin{tikzcd}
\rho_{m+1} \arrow[rrrrr, dashed, "F_{m+1}"] \arrow[d, dashed] & & & & & \rho'_{m+1} \arrow[d, dashed] \\
\rho_m \arrow[r] \arrow[d] & \rho^1_m \arrow[r] \arrow[d] & \rho^2_m \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \rho^q_m \arrow[r] \arrow[d] & \rho'_m \arrow[d] \\
\rho_{m-1} \arrow[r] \arrow[d] & \rho^1_{m-1} \arrow[r] \arrow[d] & \rho^2_{m-1} \arrow[r] \arrow[d] & \arrow[r] & \rho^q_{m-1} \arrow[r] \arrow[d] & \rho'_{m-1} \arrow[d] \\
\rho_{m-2} & {} & {} & & {} & \rho'_{m-2}
\end{tikzcd}$$

$\rho/_m$ $\qquad\qquad$ $\rho'/_m$

Let $[a_1^{m+1}] \ldots\ldots [a_r^{m+1}]$ be a preferred basis for ρ_{m+1} and $[b_1^{m+1}]. . . [b_r^{m+1}]$ a preferred basis for ρ'_{m+1} such that $F_{m+1}[a_i^{m+1}] = [b_i^{m+1}]$. Suppose first that $G^1 : \rho/_m \to \rho^1$ is an elementary contraction. Define a homotopy system $\hat{\rho}^1$ extending ρ^1 by one dimension as follows. Let $(\hat{\rho}^1)_{m+1}$ be the free $Z[\pi_1]$-module with formal generators $[a_1^{m+1}] \ldots [a_r^{m+1}]$, and define $d^{\hat{\rho}^1}_{m+1}\,[a_i^{m+1}] = G^1_m \circ d^{\rho}_{m+1}\,[a_i^{m+1}]$. Let

$\hat{G}^1_{m+1} : \rho_{m+1} \to \hat{\rho}^1_{m+1}$ be the $Z[\pi_1]$- module homomorphism

induced by the assignment $\hat{G}^1_{m+1}\,[a_i^{m+1}] = [a_i^{m+1}]$. Then

$\hat{G}^1 : \rho/_{m+1} \to \hat{\rho}^1$ is a homomorphism, and is an elementary contraction of the same dimension as G^1.

If G^1 is a based isomorphism or an elementary expansion of dimension $\leq m$, we define $\hat{G}^1 : \rho/_{m+1} \to \hat{\rho}^1$ in the same way. If G^1 is an elementary expansion of dimension (m+1) we treat the added generator of ρ^1_{m+1} as an additional formal generator of $\hat{\rho}^1_{m+1}$ and the same procedure applies. Inductively, we extend ρ^1 to a homotopy system $\hat{\rho}^1$ of dimension n, and extend G^1 to an elementary equivalence $\hat{G}^1 : \rho \to \hat{\rho}^1$ of dimension $\leq m+1$. Continuing we extend $G^2 : \rho^2 \to \rho^3$. Suppose $G^{j+1} : \rho^j \to \rho^{j+1}$ is an elementary contraction of dimension (m+1). The sub-module of $\hat{\rho}^j_{m+1}$ generated by $[a_1^{m+1}], \ldots\ldots, [a_r^{m+1}]$ has intersection O with ρ^j_{m+1} so this step also extends in the same way.

The last step $G^{q+1} : \rho^q \to \rho'/_m$ is somewhat different since the boundary $d^{\rho'}_{m+1}$ is already defined. We can assume G^{q+1} is a based isomorphism (if not, we add an additional step). Therefore $\rho^q_{m+1} = 0$, i.e. any additional generators in dimension m+1 that were introduced along the way by $G^1, \dots G^q$ have now been eliminated. Therefore $\hat{\rho}^q_{m+1}$ has generators $[a^{m+1}_1], \dots [a^{m+1}_r]$, and we define $\hat{G}^{q+1}[a^{m+1}_i] = [b^{m+1}_i]$. We have to check that the diagram commutes.

$$\begin{tikzcd}
\hat{\rho}^q_{m+1} \arrow[r, "\hat{G}^{q+1}_{m+1}"] \arrow[d] & \rho'_{m+1} \arrow[d] \\
\rho^q_m \arrow[r, "G^{q+1}_m"] & \rho'_m
\end{tikzcd}$$

We have $F_m \circ d_{m+1} = d'_{m+1} \circ F_{m+1}$ and

$$\hat{G}^{q+1}_{m+1} \circ \hat{G}^q_{m+1} \circ \dots \circ \hat{G}^1_{m+1} [a^{m+1}_i] = [b^{m+1}_i] = F_{m+1}[a^{m+1}_i]$$

and $G^{q+1}_m \circ G^q_m \dots \circ G^1_m = F_m$. Inductively all the previous squares commute

$$\begin{tikzcd}
\hat{\rho}^{j}_{m+1} \arrow[r, "\hat{G}^{j+1}_{m+1}"] \arrow[d] & \hat{\rho}^{j+1}_{m+1} \arrow[d] \\
\rho^{j}_{m} \arrow[r, "G^{j+1}_{m}"] & \rho^{j+1}_{m}
\end{tikzcd}$$

$$\begin{aligned}
\text{so } F_m \circ d_{m+1} [a_i^{m+1}] &= G_m^{q+1} \circ G_m^{q} \circ \ldots\ldots \circ G_m^{1} \circ d_{m+1} [a_i^{m+1}] \\
&= G_m^{q+1} \circ \ldots\ldots \circ G_m^{2} \circ \hat{d}^{1}_{m+1} \circ \hat{G}^{1}_{m+1} [a_i^{m+1}] \\
&\ \ \vdots \\
&= G_m^{q+1} \ \hat{d}^{q}_{m+1} [a_i^{m+1}] \\
&= d'_{m+1} \circ F_{m+1} [a_i^{m+1}] \\
&= d'_{m+1} [b_i^{m+1}].
\end{aligned}$$

Thus $G_m^{q+1} \ \hat{d}^{q}_{m+1} [a_i^{m+1}] = d'_{m+1} \ \hat{G}^{q+1}_{m+1} [a_i^{m+1}]$ as required. The same argument works in higher dimensions. This completes the proof of the lemma. □

We now prove the main theorem of this Chapter (3.5). For convenience we first re-state it in somewhat greater detail.

3.18) <u>Theorem.</u> Let M^n be a connected closed manifold, $n \geq 9$. Let x_0, x'_0 be chosen base points, $\mathcal{H}$ a handle decomposition of (M, x_0, x'_0), and $\pi_1 = \pi_1(M, x_0)$. Let $\mathcal{C}$ be a based $Z[\pi_1]$ complex of dimension n such that both $\mathcal{C}$ and the dual chain complex $\mathcal{D}(\mathcal{C})$ admit homotopy groups which we will write as $h: \rho \to \mathcal{C}$ and $h': \rho' \to \mathcal{D}(\mathcal{C})$. Then $\mathcal{C}$ is realized by a handle decomposition $\mathcal{H}_{\mathcal{C}}$ of (M, x_0, x_0') if and only if there exists a simple chain equivalence $G: \breve{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$. (+) Moreover, $\mathcal{H}_{\mathcal{C}}$ can be chosen so that $\rho(M, \mathcal{H}_{\mathcal{C}}) \approx \rho$ and $\rho(M, (\mathcal{H}_{\mathcal{C}})') \approx \rho'$. If $F: \rho(M, \mathcal{H}) \to \rho$ and $F': \rho' \to \rho(M, \mathcal{H}')$ are lifts of G and $\mathcal{D}(G)$ respectively, then there exists a diffeomorphism $g: M \to M$ isotopic to the identity, such that $g \in T_{\mathcal{H}, \mathcal{H}_{\mathcal{C}}}$ and $g_{\#}$ realizes F, and $(g^{-1})_{\#}$ realizes F'.

<u>Proof.</u> Let $F: \rho(M, \mathcal{H}) \to \rho$ be a lift of G.

$$\begin{tikzcd}
\rho(M, \mathcal{H}) \arrow[r, "F"] \arrow[d] & \rho \arrow[d] \\
\breve{\mathcal{C}}(M, \mathcal{H}) \arrow[r, "G"] & \mathcal{C}
\end{tikzcd}$$

By (3.7) F is a simple equivalence. By the Rearrangement Lemma (3.12), there exist simple equivalences F^1, F^2, $F \simeq F^2 \circ F^1$

$$\rho(M, \mathcal{H}) \xrightarrow{F^1} \rho^1 \xrightarrow{F^2} \rho \quad .$$

(+) Here G is a chain map in the restricted sense i.e. G is associated with the identity on π_1 and any lift of G is monic (see remaris preceding (3.14) above).

F^1 is a composition of based isomorphisms and elementary equivalences of dimension $\leq$ n-3. F^2 is a composition of a sequence of based isomorphisms and elementary equivalences of dimension ≥ 6: it is not hard to see that such a sequence can be rearranged to include only a single based isomorphism L occuring at the end. We will assume this has been done so that we have

$$\rho(M, \mathcal{H}) \xrightarrow{F^1} \rho^1 \xrightarrow{F^2} \rho^2 \xrightarrow[\approx]{L} \rho .$$

Applying (3.13) to F^1 there exists a handle decomposition $\mathcal{H}^1$ of (M, x_0, x_0) such that $\rho(M, \mathcal{H}^1) \approx \rho^1$. By the proof of (3.12) we can assume F^1 is monic, so there exists a diffeomorphism $f_1 \in T_{\mathcal{H}, \mathcal{H}^1}$ isotopic to the identity such that $(f_1)_\#: \rho(M, \mathcal{H}) \to \rho(M, \mathcal{H}^1)$ realizes F^1.

Although F, F^1 and L are all monic, F^2 might not be. Since F^1 is a composition of standard elementary equivalences of low dimension it has a chain homotopy inverse $(F^1)'$ which is monic. Therefore $L^{-1} \circ F \circ (F^1)' \underset{\xi}{=} F^2$. Let $F^3: \rho^1 \to \rho^2$ be defined by $F^3_n = \xi_n d_n + F^2_n$, $F^3_{n-1} = d_n \xi_n + F^2_{n-1}$ and $F^3_j = F^2_j$ for $j \leq n-2$. Then F^3 is monic and $F = L \circ F^3 \circ F^1$.

To continue we first project down to the chain level and then dualize. Let F^4 be a lift of $\mathcal{D}((L \circ F^3)_*)$.

$$\begin{tikzcd}
\rho(M, \mathcal{H}^1) \arrow[r, "L \circ F^3"] \arrow[d] & \rho \arrow[d] \\
\breve{\mathcal{C}}(M, \mathcal{H}^1) \arrow[r, "(L \circ F^3)_*"] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\breve{\mathcal{C}}(M, (\mathcal{H}^1)') & \mathcal{D}(\mathcal{C}) \arrow[l, "\mathcal{D}((L \circ F^3)_*)"'] \\
\rho(M, (\mathcal{H}^1)') \arrow[u] & \rho' \arrow[l, "F^4"'] \arrow[u]
\end{tikzcd}$$

By (3.15) and (3.7) F^4 is a simple equivalence. Now F^2 was a composition of standard elementary equivalences all of dimension ≥ 6, and $F_j^3 = F_j^2$ for $j \leq n-2$, so with respect to some families of preferred bases the matrix representing $(L \circ F^3)_{*\,i}$ is the identity for $i \leq 4$. Therefore, with respect to the dual preferred bases the matrix representing $\mathcal{D}((L \circ F^3)_*)_i$ is the identity for $i \geq n-4$. By (3.16) and (3.17) there is a sequence of elementary equivalences $H^1, \ldots H^k$ such that $F^4 = H^k \circ \ldots \circ H^1$ and dimension $H^t \leq n-4$, $1 \leq t \leq k$. Applying (3.13) to the H^t there exists a handle decomposition $\mathcal{H}^3$ such that $\rho(M, (\mathcal{H}^3)') \approx \rho'$.

Therefore $\breve{\mathcal{C}}(M, (\mathcal{H}^3)') \cong \mathcal{O}(\mathcal{C})$ and $\breve{\mathcal{C}}(M, \mathcal{H}^3) \cong \mathcal{C}$. All the elementary equivalences H^t had dimension $\leq n-4$, so the effect on $\rho(M, \mathcal{H}^1)$ of realizing them by (3.13) involves adding and cancelling pairs of handles of dimension $\geq (5, 4)$. Therefore the homotopy system $\rho(M, \mathcal{H}^1)$ is unchanged in the bottom dimensions (≤ 3) by this process i.e. $\rho(M, \mathcal{H}^3) \approx \rho$. Thus $\mathcal{H}^3$ is the required handle decomposition of (M, x_0, x_0').

To obtain the diffeomorphism $g \in T_{\mathcal{H}, \mathcal{H}^3}$ observe that by (3.13) there exists a diffeomorphism f_2 isotopic to the identity such that $f_2 \in T_{(\mathcal{H}^3)', (\mathcal{H}^1)'}$ and $(f_2)_\#$ realizes F^4. Therefore $f_2^{-1} \in T_{\mathcal{H}^1, \mathcal{H}^3}$ and $(f_2^{-1})_* : \breve{\mathcal{C}}(M, \mathcal{H}^1) \to \breve{\mathcal{C}}(M, \mathcal{H}^3)$ realizes $(L \circ F^3)_*$. Hence $(f_2^{-1} \circ f_1) \in T_{\mathcal{H}, \mathcal{H}^3}$ and $(f_2^{-1} \circ f_1)_* : \breve{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$ realizes $(L \circ F^3 \circ F^1)_* = G$. It follows from the results of Chapter Two that there exists $g \in T_{\mathcal{H}, \mathcal{H}^3}$ isotopic to $(f_2^{-1} \circ f_1)$ and hence to the identity, such that $g_\#$ realizes F and $(g^{-1})_\#$ realizes F'. This completes the proof. $\square$

Conclusions

Recall Problem II from the Introduction: What is the minimum topological entropy of a fitted diffeomorphism in a component of $\mathrm{Diff}^r(M)$? We now show how our results reduce this to an algebraic problem. Our presentation follows that of Shub in [19].

4.1) Definition. Let (M^n, x_0, x_0') and $\mathcal{H}$ be as in Theorem (3.18). That is , M^n is a closed connected manifold of dimension ≥ 9, x_0 and x_0' are base points, and $\mathcal{H}$ is a handle decomposition of (M, x_0, x_0'). Let $\pi_1 = \pi_1(M, x_0)$ and let $\mathcal{C}$ be a based $Z[\pi_1]$ complex such that both $\mathcal{C}$ and $\mathcal{D}(\mathcal{C})$ admit homotopy groups; $h: \rho \to \mathcal{C}$ and $h': \rho' \to \mathcal{D}(\mathcal{C})$. We will say that the triple $(\mathcal{C}, \rho, \rho')$ is a complex of M if and only if there exists a simple chain equivalence $G: \mathcal{C}(M, \mathcal{H}) \to \mathcal{C}$. Thus Theorem (3.18) says that the complexes of M are exactly those realized by handle decompositions of (M, x_0, x_0').

Let $E: \rho \to \rho$ and $E': \rho' \to \rho'$ be homomorphisms such that $\mathcal{D}(E) = (E')_*$. If $f: M \to M$ is a diffeomorphism in $T_{\mathcal{H}}$ we will say that the pair (E, E') is an endomorphism of f on the complex $(\mathcal{C}, \rho, \rho')$ if and only if there is a simple chain equivalence $G: \mathcal{C}(M, \mathcal{H}) \to \mathcal{C}$ such that the following diagram commutes up to chain homotopy:

$$\begin{tikzcd}
\tilde{\mathcal{C}}(M, \mathcal{H}) \arrow[r, "f_*"] \arrow[d, "G"] & \tilde{\mathcal{C}}(M, \mathcal{H}) \arrow[d, "G"] \\
\mathcal{C} \arrow[r, "E_*"] & \mathcal{C}
\end{tikzcd}$$

$\square$

4.2) Definition. Let ρ be a based homotopy system. If $E : \rho \to \rho$ is a homomorphism we define the absolute value of algebraic intersections as follows. For $k \geq 3$ we assume $E_k : \rho_k \to \rho_k$ is given as matrices over $Z[\pi_1]$, (E^k_{ij}), so $|E^k| = (|E^k_{ij}|)$. Let $\{[a^2_1], \ldots [a^2_\ell]\}$ be a preferred basis for ρ_2. We assume $E_2 : \rho_2 \to \rho_2$ is given as a list of words $E_2[a^2_i] = \epsilon_1 x_1[a^2_{i_1}] + \cdots + \epsilon_k x_k[a^2_{i_k}]$ where $\epsilon_s = \pm 1$ and $x_s \in \rho_1$. Then we associate a non-negative integral matrix $|E^2|$ by counting the occurances of $[a^2_j]$ in the given word for Spelling $E_2[a^2_i]$ that is

$$|E^2|_{i,j} = \sum_{i_s = j} |\epsilon_s| \,.$$

Similarly, let $\{[a^1_1], \ldots\ldots, [a^1_m]\}$ be a preferred basis of ρ_1. We assume $E_1 : \rho_1 \to \rho_1$ is given as a list of words

$$E_1[a^1_i] = [a^1_{i_1}]^{\epsilon_1} \cdots [a^1_{i_\ell}]^{\epsilon_\ell}$$

and define $|E^1|_{i,j} = \sum_{i_s = j} |\epsilon_s|$.

Thus, in each dimension $k=1,\ldots, n-1$, we associate a non-negative integral matrix $|E^k|$ with $E_k: \rho_k \to \rho_k$. Let

$s(|E|) = \max_k \; s(|E^k|)$. □

4.3) Theorem. Let (M, x_0, x_0') and $\mathcal{H}$ be as above (4.1). Let $f \in T_{\mathcal{H}}$ and suppose (E, E') is an endomorphism of f on the complex of M $(\mathcal{C}, \rho, \rho')$. Then f is isotopic to a fitted diffeomorphism g such that

$$h(g) = \max \{ \log \; s(|E|), \; \log \; s(|E'|)\} .$$

Proof. Let $G: \tilde{\mathcal{C}}(M, \mathcal{H}) \to \mathcal{C}$ be a simple chain equivalence such that $G f_* \simeq E_* G$, and let $F: \mathcal{C} \to \tilde{\mathcal{C}}(M, \mathcal{H})$ be a chain homotopy inverse for G. By (3.18) there exists a handle decomposition $\mathcal{H}^1$ of (M, x_0, x_0') which realizes $(\mathcal{C}, \rho, \rho')$ and diffeomorphisms $\ell, k: M \to M$, isotopic to the identity, such that $\ell \in T_{\mathcal{H}, \mathcal{H}^1}$, ℓ_* realizes G, $k \in T_{\mathcal{H}^1, \mathcal{H}}$ and k_* realizes F. Therefore f is isotopic to $\ell \circ f \circ k \in T_{\mathcal{H}^1}$. Since $G f_* \simeq E_* G$, $G f_* F \simeq E_* GF \simeq E_*$. Therefore $(\ell \circ f \circ k)_* \simeq E_*$. By the results of Chapter Two (2.15) $(\ell \circ f \circ k)$ is isotopic to a diffeomorphism $g \in T_{\mathcal{H}^1}$ such that $g_\# = E$ and $(g^{-1})_\# = E'$. When g is fitted the algebraic intersections are unchanged, so by the results of Chapter One we obtain finally a diffeomorphism g fitted with respect to $\mathcal{H}^1$ and isotopic to f, such that

$h(g) = \max \{\log \; s(|E|), \; \log \; s(|E'|)\}$. □

Let (M, x_0, x_0') and $\mathcal{U}$ be as above. If $f: M \to M$ is a diffeomorphism and $f \in T_{\mathcal{U}}$ then Theorem (4.3) reduces to algebra the problem of finding the minimum entropy of diffeomorphisms isotopic to f and fitted with respect to some handle decomposition of (M, x_0, x_0'). If $f \in \mathrm{Diff}^r(M)$ is an arbitrary diffeomorphism we pick some handle decomposition $\mathcal{U}$ and deform f in any way to $f_1 \in T_{\mathcal{U}}$. When we apply (4.3) the entropy detected will be independent of which handle decomposition $\mathcal{U}$ we picked, and of how we made f preserve it.

This completes our discussion. $\square$

References

[1] R. L. Adler, M. G. Konheim and M. H. McAndrew, Topological entropy. Trans. Amer. Math Soc. 114 (1965) 309-319.

[2] R. Bowen, Topological entropy and Axiom A. Proc. Symp. Pure Math. Vol XIV Global Analysis.

[3] R. Bowen, Entropy versus homology for certain diffeomorphisms. Topology Vol 13 (1974) 61-67.

[4] T. Chapman, Topological invariance of Whitehead torsion. Amer. Journal of Math. Vol 96 (1974) 488-497.

[5] M. M. Cohen, A Course in Simple Homotopy Theory. Springer Verlag (1970).

[6] A. Haefliger, Plongements differentiables de variétés dans variétés. Comment. Math. Helvet. Vol 36 (1961) 47-82.

[7] A. Haefliger, Plongements de variétés dans le domaine stable. Sem. Bourbaki Exposé 245 1-15.

[8] M. Kervaire, La theor[illegible] de Barden-Mazur-Stallings. Comment. Math. Helvet. [illegible] (1965) 31-42.

[9] A. Manning, Topological entropy and the first homology group. Proc. Warwick Dynamical Systems 1974 Springer Verlag (1975).

[10] J. Milnor, Lectures on the h-cobordism theorem. Princeton Univ. Press (1965).

[11] J. Milnor, Differentiable Structures. Mimeographed lecture notes. Princeton Univ. (1961).

[12] J. Milnor, Whitehead torsion. Bull. A.M.S. Vol 72 (1966) 358-426.

[13] J. Milnor, A duality theorem for Reidemeister torsion. Ann. Math. 76 (1962) 137-147.

[14] J. Robbin, On structural stability. Ann. Math. 94 (1971).

[15] C. P. Rourke and B. J. Sanderson, Introduction to P. L. Topology. Springer Verlag (1972).

[16] M. Shub and D. Sullivan, Homology theory and dynamical systems. Topology Vol 14 (1975) 109-132.

[17] M. Shub and R. Williams, Entropy and Stability. Topology Vol 14 (1975) p. 329.

[18] M. Shub, Structurally stable diffeomorphisms are dense. Bull. Amer. Math. Soc. Vol 78 (1972) p. 817.

[19] M. Shub, Dynamical systems, filtrations, and entropy. Bull. Amer. Math. Soc. Vol 80 (1974) p. 27.

[20] M. Shub, Morse-Smale diffeomorphisms are unipotent on homology. Proc. Symp. Dynamical Systems Salvador (1971).

[21] S. Smale, Stability and isotopy in discrete dynamical systems. Proc. Symp. Dynamical Systems Salvador (1971).

[22] S. Smale, On the structure of manifolds. Amer. Journal of Math. Vol 84 (1962) 387-399.

[23] E. H. Spanier, Algebraic Topology. McGraw-Hill (1966).

[24] C. T. C. Wall, Formal Deformations. Proc. London Math. Soc. (3) 16 (1966) 342-52.

[25] J. H. C. Whitehead, Combinatorial Homotopy II. Bull. Amer. Math. Soc. Vol 55 (1949) 453-496.

[26] J. H. C. Whitehead, Simple Homotopy Types. Amer. Journal of Math. Vol 72 (1950). 1-57.

[27] J. H. C. Whitehead, On incidence matrices, nuclei, and homotopy types. Ann. Math. 42 (1941) 1197-1239.

[28] C. Pugh, On the entropy conjecture. Proc. Warwick Dynamical Systems 1974 Springer Verlag (1975).

[29] F. Gantmacher, The Theory of Matrices. GITTL, Moscow (1953). English translation Chelsea, New York, (1959).